Application of Volume & Salt Conservation: Estuarine Circulation

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The most essential use of the equations of motion of a fluid is to calculate budgets over a known volume or mass of fluid.

Perhaps the nicest example of this in oceanography is the flow into and out of estuaries. This estuarine circulation is a useful example of the use of volume and salt conservation attributed to *Knudsen* (1900).

Here, we will:

1. derive the budgets for volume and salt conservation,
2. then do examples from the Black and Mediterranean Sea (from *Pickard and Emery*, 1990),
3. consider pollutants along with salt,
4. and finally solve a time-dependent problem to demonstrate the role of 'the flushing timescale'.

We begin with the conservation of mass*:

$$\frac{d}{dt}\int_{V(t)} \rho \, \mathrm{d}V = 0 \rightarrow \frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{v} = 0. \quad (1)$$

Which is replaced by conservation of volume (for an incompressible fluid):

$$\nabla\cdot\mathbf{v} = 0. \quad (2)$$

*Where we recall our notation that $V(t)$ follows a material surface.

Conservation of salt is (neglecting molecular diffusion):

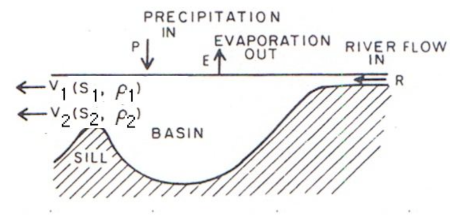
$$\frac{d}{dt}\int_{V(t)} \rho S \, \mathrm{d}V = \int_{V(t)} \rho \frac{DS}{Dt} \, \mathrm{d}V = 0. \qquad (3)$$

Or, since it didn't matter which parcel of fluid we followed and $\rho > 0$:

$$\frac{DS}{Dt} = 0. \qquad (4)$$

We note that salinity can change (e.g., by evap./precip.), but it is the quantity of water that changes not the mass of salt.

The following figure schematizes the situation (*Pickard and Emery*, 1990). There is a two-layer flow over the sill, runoff, evaporation and precipitation.



We proceed by integrating the differential equations over the estuary.

$$\nabla \cdot \mathbf{v} = 0 \qquad (5)$$

applies everywhere, so

$$0 = \int \nabla \cdot \mathbf{v} \, \mathrm{d}V = \oint \mathbf{v} \cdot \hat{\mathbf{n}} \, \mathrm{d}A. \qquad (6)$$

Where $\int \, \mathrm{d}V$ is by Gauss's identity is equivalent to $\oint \, \mathrm{d}A$ over the surrounding surface. We break up the surface integral:

$$\begin{aligned} 0 &= \oint \mathbf{v} \cdot \hat{\mathbf{n}} \, \mathrm{d}A, & (7) \\ &= \int_{A_1} \mathbf{v_1} \, \mathrm{d}A + \int_{A_2} \mathbf{v_2} \, \mathrm{d}A, & (8) \\ &\quad + \int_{A_E} E \, \mathrm{d}A - \int_{A_P} P \, \mathrm{d}A + \int_{A_R} \mathbf{v_r} \, \mathrm{d}A. \end{aligned}$$

Renaming using volume fluxes $[V] = L^3/T$:

$$\begin{aligned} 0 &= \int_{A_1} \mathbf{v_1} \, \mathrm{d}A + \int_{A_2} \mathbf{v_2} \, \mathrm{d}A, & (9) \\ &\quad + \int_{A_E} E \, \mathrm{d}A - \int_{A_P} P \, \mathrm{d}A + \int_{A_R} \mathbf{v_r} \, \mathrm{d}A. \\ 0 &= V_1 + V_2 + A_E E - A_P P - R, & (10) \\ &\equiv V_1 + V_2 - F. & (11) \end{aligned}$$

Where F is the freshwater supplied to the estuary.

Recall: conservation of salt gives the equation on salinity (neglecting mixing/diffusion):

$$0 = \frac{DS}{Dt} = \frac{\partial S}{\partial t} + \mathbf{v} \cdot \nabla S.$$

We use $\nabla \cdot \mathbf{v} = 0$, to find

$$0 = \frac{\partial S}{\partial t} + \nabla \cdot \mathbf{v} S. \tag{12}$$

Integrate to find

$$0 = \int \nabla \cdot \mathbf{v} S \;\; \mathrm{d}V = \oint S \mathbf{v} \cdot \hat{\mathbf{n}} \; \mathrm{d}A. \tag{13}$$

Little salt in rivers, evap & precip. carry no salt, so

$$0 = \oint S \mathbf{v} \cdot \hat{\mathbf{n}} \; \mathrm{d}A = \int_{A_1} S \mathbf{v} \cdot \hat{\mathbf{n}} \; \mathrm{d}A + \int_{A_2} S \mathbf{v} \cdot \hat{\mathbf{n}} \; \mathrm{d}A.$$

Formally, we can define velocity-weighted average salinities, so

$$\begin{aligned} 0 &= \int_{A_1} S \mathbf{v} \cdot \hat{\mathbf{n}} \; \mathrm{d}A + \int_{A_2} S \mathbf{v} \cdot \hat{\mathbf{n}} \; \mathrm{d}A, \\ &= S_1 V_1 + S_2 V_2. \end{aligned}$$

Where we define

$$S_1 \equiv \frac{\int_{A_1} S \mathbf{v} \cdot \hat{\mathbf{n}} \; \mathrm{d}A}{\int_{A_1} \mathbf{v} \cdot \hat{\mathbf{n}} \; \mathrm{d}A}.$$

But, more loosely, there is a typical incoming salinity and a typical outgoing salinity, which may make approximate values for S_1 and S_2 obvious.

So, we have two equations so far

$$S_1 V_1 = -S_2 V_2, \quad V_1 + V_2 = F.$$

We can eliminate either V_1 or V_2 using

$$V_1 = \frac{-S_2 V_2}{S_1}, \quad V_2 = \frac{-S_1 V_1}{S_2}.$$

To give

$$F = V_2 \frac{S_1 - S_2}{S_1}, \quad F = V_1 \frac{S_2 - S_1}{S_2}. \tag{14}$$

We will now do two classic examples from *Pickard and Emery* (1990): the Mediterranean and the Black Sea.

The Mediterranean has a sill depth (at the Strait of Gibraltar) of 330m. It is observed that

$$\begin{aligned} S_1 &= 36.3 \text{ psu}, \\ S_2 &= 37.8 \text{ psu}, \\ V_1 &= -1.75 \cdot 10^6 \text{ m}^3/\text{s} \equiv -1.75 \text{ Sv}. \end{aligned}$$

Where the Sverdrup, 10^6 m^3/s $\equiv 1$ Sv, is a useful oceanographic unit.

So, we can infer from

$$\begin{aligned} S_1 &= 36.3 \text{ psu}, \qquad S_2 = 37.8 \text{ psu}, \\ V_1 &= -1.75 \text{ Sv}, \\ V_2 &= \frac{-S_1 V_1}{S_2}, \qquad F = V_1 \frac{S_2 - S_1}{S_2}, \end{aligned}$$

that

$$V_2 = 1.68 \text{ Sv}, \quad F = -0.07 \text{ Sv}.$$

So, for a tiny amount of net freshwater loss (through evaporation exceeding precipitation and runoff), a huge exchange flow is required, with an outflow of salty water exiting at depth and fresher Atlantic water entering at the surface.

The Black Sea has a sill depth (at the Bosphorus) of 70m. It is observed that

$$\begin{aligned} S_1 &= 17 \text{ psu}, \\ S_2 &= 35 \text{ psu}, \\ V_1 &= 13 \cdot 10^3 \text{ m}^3/\text{s}. \end{aligned}$$

Where the Sverdrup, 10^6 m^3/s $\equiv 1$ Sv, is a useful oceanographic unit. So, we can infer that

$$V_2 = -6 \cdot 10^3 \text{ m}^3/\text{s}, \quad F = 7 \cdot 10^3 \text{ m}^3/\text{s}.$$

Compare the two basins:
Mediterranean:

$$S_1 = 36.3 \text{ psu}, S_2 = 37.8 \text{ psu}$$
$$V_1 = -1.75 \cdot 10^6 \text{ m}^3/\text{s}, V_2 = 1.68 \cdot 10^6 \text{ m}^3/\text{s},$$
$$F = -7 \cdot 10^4 \text{ m}^3/\text{s}.$$

Black:

$$S_1 = 17 \text{ psu}, S_2 = 35 \text{ psu}$$
$$V_1 = 13 \cdot 10^3 \text{ m}^3/\text{s}, V_2 = -6 \cdot 10^3 \text{ m}^3/\text{s},$$
$$F = 7 \cdot 10^3 \text{ m}^3/\text{s}.$$

Mediterranean has outflow at depth, 25x the volume of freshwater. Black has inflow at depth, 1x the volume of freshwater.

Key differences: the amount of mixing in basin (Med. has $S_1 \approx S_2$, while Black has $S_1 \ll S_2$), and inflow/outflow at surface governed by freshwater deficit/supply (assuming $S_1 < S_2$).

We could also treat pollutants, not just salt.

$$\frac{DP}{Dt} \approx 0. \tag{15}$$

The pollutants will have river sources, and potentially an exchange at sill, so the steady equation is

$$V_1 P_1 + V_2 P_2 = R P_2.$$

Suppose we have constant pollutant concentration in river discharge in Black Sea, then

$$F \approx R = 7 \cdot 10^3 \text{ m}^3/\text{s}, \quad V_1 \approx 2R,$$
$$P_2 \approx 0.$$

So, the steady state result will be

$$V_1 P_1 + V_2 P_2 = R P_2,$$
$$P_1 \approx \frac{1}{2} P_R.$$

Thus, the incoming pollution is diluted very little in the Black Sea.

If we follow the pollutant through the Bosphorus out into the Mediterranean, we have Black Sea sources:

$$P_B \approx \frac{1}{2} P_R, \qquad V_B = 13 \cdot 10^3 \text{ m}^3/\text{s}$$

And the Med. budget will be

$$V_1 P_1 + V_2 P_2 = V_B P_B \approx V_B \frac{1}{2} P_R.$$

Using our Med. numbers

$$V_1 = -1.75 \cdot 10^6 \text{ m}^3/\text{s}, V_2 = 1.68 \cdot 10^6 \text{ m}^3/\text{s},$$
$$P_1 = 0.$$

We find the Med outflow is very dilute

$$P_2 \approx \frac{1}{260} P_R.$$

Reinforcing our notion that the Med. is better mixed than the Black Sea.

One trick we haven't used is time dependence. While volume conservation didn't have a time derivative, pollutant/salt did.

Suppose we impusively started polluting the rivers, then

$$\frac{DP}{Dt} = 0 \rightarrow \frac{d}{dt} \int P \text{ d}V = -\oint P \mathbf{v} \cdot \hat{\mathbf{n}} \text{ d}S$$

We know the steady-state solution P_{ss} (we just did it!), so let's see how long it takes to get there.

How long to reach steady state?

$$\frac{d}{dt}\int (P - P_{ss})\ \mathrm{d}V = -\oint (P - P_{ss})\mathbf{v}\cdot\hat{\mathbf{n}}\ \mathrm{d}S,$$
$$\mathrm{Vol.}\frac{d}{dt}\langle P - P_{ss}\rangle = V_R(P_R - P_{R;ss}) - V_{out}(P_{out} - P_{out;ss}),$$
$$\frac{d}{dt}\langle P - P_{ss}\rangle = -\frac{V_{out}}{\mathrm{Vol.}}(P_{out} - P_{out;ss}).$$

Angle brackets are volume averages, and Vol. is the volume. If we assume a well-mixed basin, the outflow concentration, P_{out}, will be near the average $\langle P\rangle$, so

$$\frac{d}{dt}\langle P - P_{ss}\rangle \approx -\frac{V_{out}}{\mathrm{Vol.}}\langle P - P_{ss}\rangle$$

Thus,

$$\frac{d}{dt}\langle P - P_{ss}\rangle \approx -\frac{V_{out}}{\mathrm{Vol.}}\langle P - P_{ss}\rangle \quad\rightarrow\quad \langle P - P_{ss}\rangle \approx Ae^{-t\frac{V_{out}}{\mathrm{Vol.}}}$$
$$\langle P\rangle \approx \langle P_{ss}\rangle\left(1 - e^{-t\frac{V_{out}}{\mathrm{Vol.}}}\right)$$

Which introduces the *flushing time*, $Vol./V_{out}$. In the Med. (Vol.=3.8·10^{15} m^3) it's about 70 yrs, for the Black (Vol.=$6\cdot10^{14}$ m^3), it's about 1500 yrs.

*References

Knudsen, M., Ein hydrographischer lehrsatz, *Ann. Hydrogr. Marit. Meteorol.*, *28*, 316–320, 1900.

Pickard, G. L., and W. J. Emery, *Descriptive Physical Oceanography: An Introduction*, 5th ed., Butterworth Heinemann, Oxford, 1990.