Fall 2015 GEOL0350–GeoMath Final Exam

1 Intermittent Tailspin

In *Lane et al.* (1993) temperature differences from subsequent measurements at a fixed location, but repeatedly measured, in a turbulent fluid depend on the Reynolds number (R). The following two figures show key results.

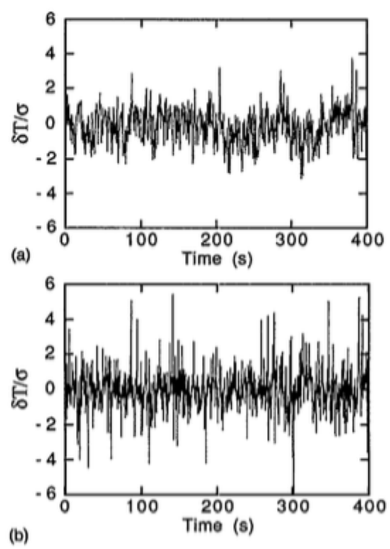

FIG. 7. Temperature time series 5 mm from the grid for (a) $R=300$; (b) $R=1850$. The temperature is expressed in units of the standard deviation. Frequent large excursions from the mean are seen in case (b).

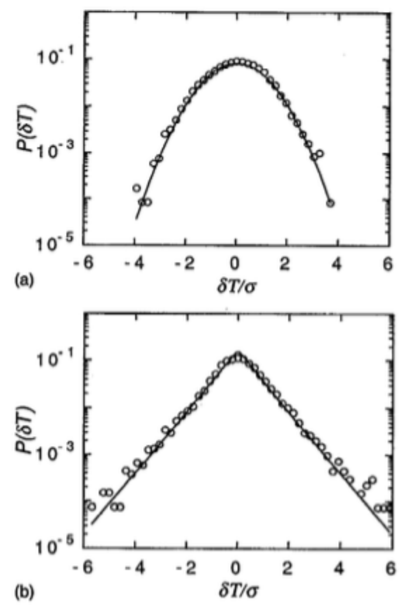

FIG. 8. Temperature probability distributions $P(\delta T)$ corresponding to Fig. 7. (a) $R=300$, (b) $R=1850$. The temperature pdf's have pronounced exponential tails at high R.

Figure 1: (a, left) Timeseries of temperature differences from mean value for R=300 run, normalized by its standard deviation. (b, left) Normalized timeseries from R=1850 run. (a, right) Probability density function (with normalized histogram) of temperature differences for R=300. (b, right) PDF and histogram for for R=1850.

Hint:

$$\int_{\mu+\sigma}^{\infty} \frac{1}{\sigma\sqrt{2}} e^{-\sqrt{2}|x-\mu|/\sigma}\,\mathrm{d}x = \frac{e^{-\sqrt{2}}}{2} \approx 0.12 \tag{1}$$

$$\int_{\mu+2\sigma}^{\infty} \frac{1}{\sigma\sqrt{2}} e^{-\sqrt{2}|x-\mu|/\sigma}\,\mathrm{d}x = \frac{e^{-2\sqrt{2}}}{2} \approx 0.03, \tag{2}$$

$$\int_{\mu+3\sigma}^{\infty} \frac{1}{\sigma\sqrt{2}} e^{-\sqrt{2}|x-\mu|/\sigma}\,\mathrm{d}x = \frac{e^{-3\sqrt{2}}}{2} \approx 0.007. \tag{3}$$

1.1 Timeseries

Compare the two timeseries, what is different about them?

The R=1850 timeseries is much more spiky, even though it has been normalized to its own standard deviation and thus the two figures have exactly the same variance. It is therefore more intermittent.

1.2 PDFs

Compare the two PDFs. What is different about them?

The R=1850 pdf is much flatter with longer tails, while the R=300 pdf appears more quadratic on semilog axes (therefore more Gaussian). R=1850 is therefore more intermittent.

1.3 And a 1, and a 2, and a . . .

The pdf of temperature for the $R = 300$ run is fit well by a normal distribution $\rho(x; \sigma, \mu) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$. As we discuss in section 14.6.4 of the notes, what percentage of the timeseries values should fall outside of $\mu \pm \sigma$, i.e., $x > \mu + \sigma$ or $x < \mu - \sigma$? What percentage are outside of $\mu \pm 2\sigma$? of $\mu \pm 3\sigma$?

Within $(1\sigma, 2\sigma, 3\sigma)$ are $68\%, 95\%, and 99.7\%$ of the data. Thus, the one-sided estimate (greater than the interval) is $1/2 - 1/2p$ where p is this probability, or $16\%, 2.5\%, and 0.15\%$. The two-sided estimates are $32\%, 5\%, and 0.3\%$.

1.4 Da Capo, Bruscamente

The pdf for the $R = 1850$ run is fit well by an exponential distribution $\rho(x; \sigma, \mu) = \frac{1}{\sigma\sqrt{2}} e^{-\sqrt{2}|x-\mu|/\sigma}$. Given the formula for integration (1) above, what percentage of the timeseries values should fall outside of $\mu \pm \sigma$, $\mu \pm 2\sigma$, $\mu \pm 3\sigma$?

Based on the integral formulas, the one-sided estimate (greater than the interval) is $12\%, 3\%, and 0.7\%$. The two-sided estimates are $24\%, 6\%, and 1.4\%$. Thus, fewer points are outside of the 1σ interval, about the same for the 2σ interval and more than four times as many outside of the 3σ interval.

1.5 Graph it!

Indicate the $\pm 3\sigma$ location on all four graphs above. Which experiment–$R = 300$ or $R = 1850$–has more of the probability outside of this range?

Drawing two horizontal line at the $\pm 3\sigma$ levels on the left column, and vertical lines on the $\pm 3\sigma$ levels on the right column reveals that the R=1350 experiment is significantly more populated outside of this range.

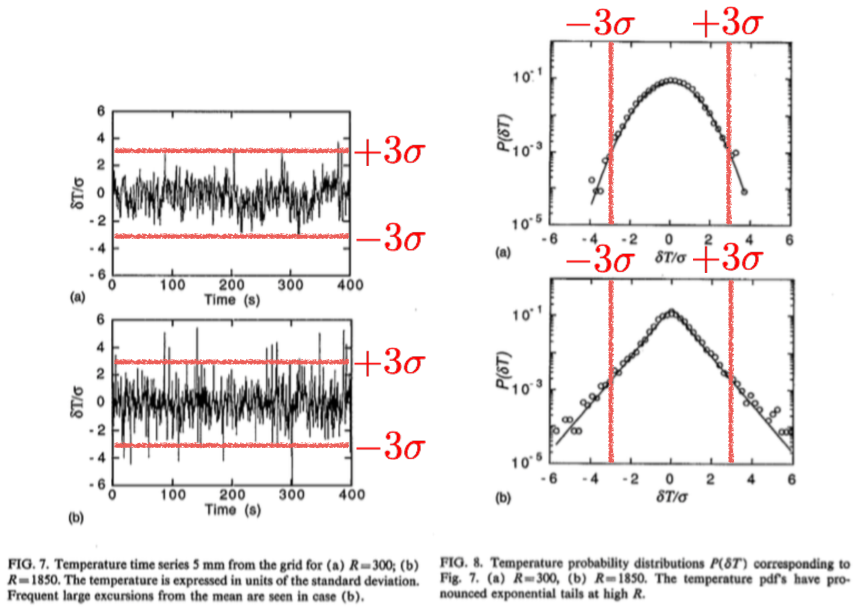

Figure 1: (a, left) Timeseries of temperature differences from mean value for R=300 run, normalized by its standard deviation. (b, left) Normalized timeseries from R=1850 run. (a, right) Probability density function (with normalized histogram) of temperature differences for R=300. (b, right) PDF and histogram for for R=1850.

2 Sedimentation Age of Aquarius

> The Age of Aquarius is an astrological term denoting either the current or forthcoming astrological age, depending on the method of calculation. Astrologers maintain that an astrological age is a product of the earth's slow precessional rotation and lasts for 2,160 years, on average (26,000-year period of precession / 12 zodiac signs = 2,160 years).

–Wikipedia

To explore the geology of the Age of Aquarius (here taken to be the last 2,160 years), we examine a sediment core. To do so, we need to figure out an age model, a relationship between the depth within the core and the age. This age model is uncertain, so we'd like to model the uncertainty as well. The top of the core is known to be the present-day, and the age model predicts a linear relationship between age and depth. We can assume the error in every slice is independent from all others.

2.1 To Get Back to Capricorn

At the constant accumulation rate we expect, our 2.16 meter core just reaches the Age of Capricorn (The age before Aquarius). What is the steady accumulation rate of sediment (meters/year)? What is its inverse (years/meter)?

The total accumulation is 2.16 m in 2160 yr, or $1m/1000yr$. This is $1000yr/m$ or $10yr/cm$.

2.2 Erroneous Astrology? No Way!

We divide the core into 100 even slices of 2.16 cm. It turns out the error in sedimentation rate works gives a Gaussian duration error of each slice with a uniform standard deviation of $\sigma = 10$yr–i.e., each 2.16 cm slice is 21.6 ± 10yr of the Age of Aquarius. What is the standard error in the average duration per 2.16 cm slice, when averaged over all of the slices?

The standard error of the average is $\sigma/\sqrt{N} = 10/\sqrt{100} = 1yr$.

2.3 Take 2

What is the expected error in age at the bottom of 2 slices? (Note that the age error accumulates with each additional slice, but it is not just the sum of the two standard deviations).

We should add the variances using error propagation summing variances to find $\sqrt{2\sigma^2} = \sqrt{2}\sigma \approx 14yr$.

2.4 Get to the bottom of this.

What is the expected error in age at the bottom–i.e., the sum of all of the slices?

There are two ways to get at this. We could multiply the standard error by the number of slices to find $N\sigma/\sqrt{N} = N/\sqrt{N}\sigma = 100yr$ or add the variances using error propagation to find $\sqrt{N\sigma^2} = \sqrt{N}\sigma = 100yr$.

2.5 Graphical

Draw a picture showing how the age error accumulates from the top of the core down to the bottom, keeping in mind that the age at the bottom of the first slice is σ, and that you have just predicted the rate at which error accumulates with depth.

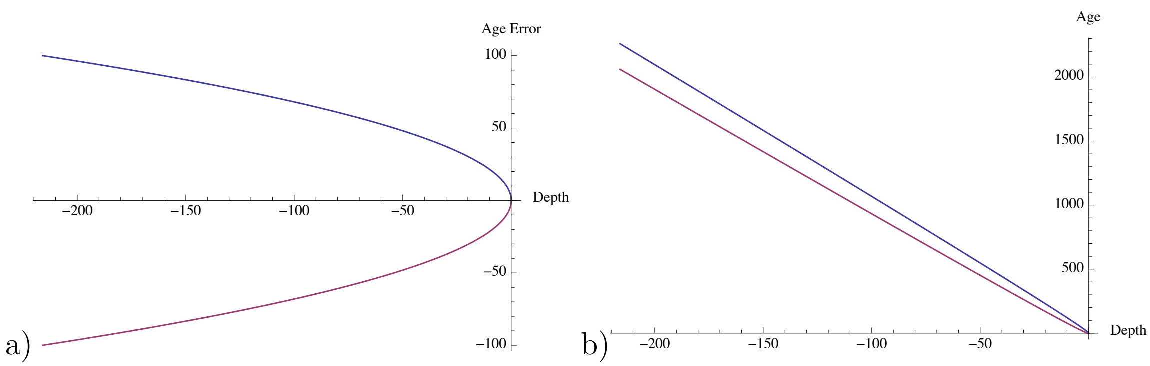

Figure 2: a) Error versus depth. b) Age with uncertainty versus depth. Either is acceptable–should indicate square root growth.

3 Helmholtz and Fourier Walk into a Bar

A warm temperature anomaly is localized in the middle of a long bar (big span in z), which is rectangular in cross-section ($|x| < W, |y| < W$). After performing separation of variables, and fitting to the initial condition, we find that the spatial pattern of the temperature field must obey Helmholtz's equation (see notes Section 10.4 if unclear):

$$\nabla^2 \phi_k(x, y, z) + k^2 \phi_k(x, y, z) = 0. \tag{4}$$

Paying attention to the boundary conditions, a solution is assumed of the form:

$$\phi_k(x, y, z) = A \cos mx \cos ny e^{pz}. \tag{5}$$

3.1 BCs

If mL and nL are both multiples of π, what are the boundary conditions in terms of $\partial\phi/\partial x$ and $\partial\phi/\partial y$ on the sides of the bar?

$\cos i\pi$ is ± 1, but the derivative of cos at these location is zero. Thus, in terms of the derivatives of ϕ, the boundary conditions are $\partial\phi/\partial n = 0$. This is an Neumann or insulating boundary condition.

3.2 Field Eq.

What is the condition on m, n, p such that the Helmholtz (field) equation is satisfied?

Plugging in, we find that

$$-m^2 - n^2 + p^2 + k^2 = 0, \tag{6}$$
$$p^2 = m^2 + n^2 - k^2. \tag{7}$$

3.3 Pint

Describe the solution if $m^2 + n^2 - k^2 > 0$.

In this case, p^2 will be positive and the shape function will decay or grow exponentially away from the heat anomaly. It will oscillate in x, y.

3.4 Shot

Describe the solution if $m^2 + n^2 - k^2 < 0$.

In this case, p^2 will be negative, so p will be imaginary and the shape function will oscillate in x, y, z.

3.5 Glass

Describe the solution if $m^2 + n^2 - k^2 = 0$.

In this case, $p^2 = 0$ so the shape function will be constant in z and oscillated in x, y.

4 Going to the UK

In honor of my sabbatical destination, I have chosen the following dynamical system (you may be able to guess why as you draw).

$$\dot{x} = (r + x)(r - x)(rx). \tag{8}$$

4.1 Keep Calm and Steady on

Find the steady solutions to the dynamical system.

$$r \neq 0 : \dot{x^*} = 0 \text{ for if one of: } x^* = r, x^* = -r, x^* = 0. \tag{9}$$

$$r = 0 : \text{ all } x \tag{10}$$

4.2 Portrait of the Artist

Sketch the phase portrait for $r = -1$ and $r = 1$ and $r = 0$ (3 graphs). Indicate the stability of any fixed points.

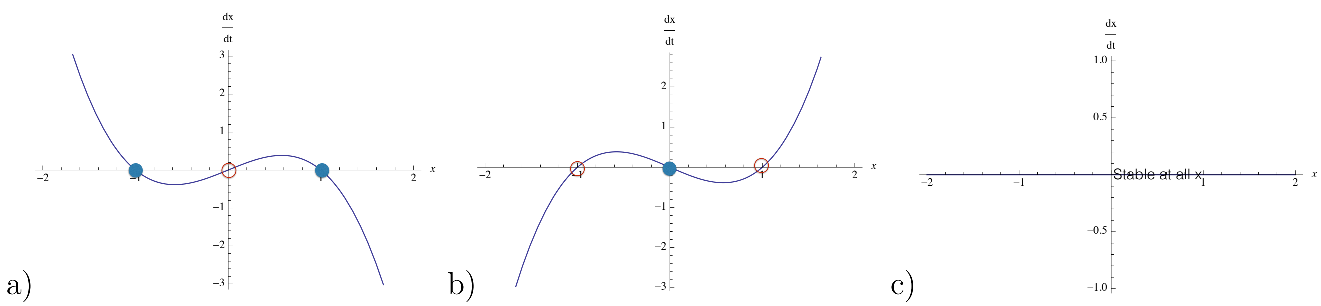

Figure 3: a) $r = 1$ b) $r = -1$. c) $r = 0$

4.3 Bi-union Jack

Sketch the bifurcation diagram for r ranging from -1 to 1, indicating stability.

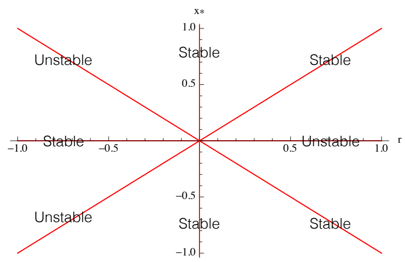

Figure 4: Bifurcation diagram (Union Jack)! For $r > 0$, the branches are (from bottom), stable, unstable, stable. For $r < 0$, the branches are unstable, stable, unstable. For r=0, all values of x are stable (vertical line).

4.4 What does the bifurcation diagram have to do with my sabbatical (bonus question)?

The bifurcation diagram is a British flag!

References

Lane, B., O. Mesquita, S. Meyers, and J. P. Gollub (1993), Probability distributions and thermal transport in a turbulent grid flow, *Physics of Fluids A: Fluid Dynamics (1989-1993)*, *5*(9), 2255–2263.