Fall 2019 GEOL0350–GeoMath Midterm Exam

1 Stones Down a Hill

Consider stones rolling down a hill from A to B, then up an opposing hill (B to C), then back down to the bottom and up toward where they begin (C to B to A).

Figure 1: **Stones that are Rolling.**

1.1 Potential Energy

The stones' potential energy ($E = mgz$) acts as a potential for the force of gravity. Take the gradient of this potential energy, and describe the force and its direction.

The gradient of the potential energy is mg, it is the gravitational force pointing downward.

1.2 Force and Work

The work done (energy released) by the force whose potential is the potential energy is $\int(\nabla mgz) \cdot \mathrm{d}\mathbf{l}$. If the stone makes the full loop from A to A, how much work is done?

According to the gradient theorem, $\int_A^A \nabla mgz \cdot \mathrm{d}\mathbf{l} = 0$.

1.3 Muddy

Now suppose that the hills are muddy, so a force opposing the direction of travel slows the stones. Sisyphus has to help out in getting the stones up the final hill by rolling them back up to A. What is the amount of work done by the *conservative* force of gravity during this whole process?

According to the gradient theorem, $\int_A^A \nabla mgz \cdot \mathrm{d}\mathbf{l} = 0$.

1.4 Nonconservative

Now, thinking about the force applied by the mud, and the work done by it ($\int \mathbf{F} \cdot \mathrm{d}\mathbf{l}$), can this force be conservative? No, because the force applied by the mud is always in the opposite direction of the motion, so its integral is negative, which means it cannot be represented by a gradient.

1.5 Frictional Function

If Sisyphus pushes the stones up one path back to A one time, and then up a different path to A the next time, will the work done by the conservative force differ? Will the work done by the non-conservative force differ?

No, the gradient theorem is path-independent. Maybe, it is possible that the non-conservative forces happen to be the same, but they are not required to be.

2 Deep Time

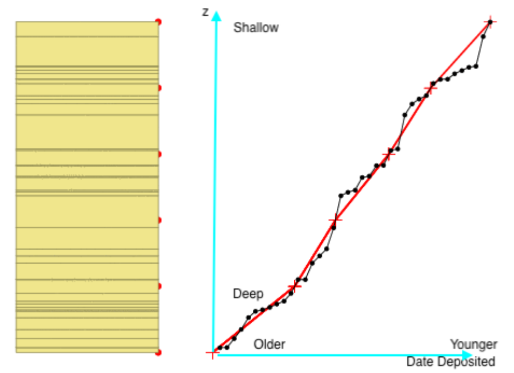

Figure 2: **A sediment core and time of deposition vs. depth.**

An interesting geological problem is relating the depth of a sediment to its age. The figure illustrates the sediment (with laminated layers) and the accumulation of its depth by date of deposition and a fit.

2.1 Linear?

Suppose we write the height of the sediment pile z as a function of the year it was deposited t, with a constant rate r. Then,

$$z = r(t - t_0),$$

Where t_0 is the year when depth $z = 0$ was deposited.
Consider a time t_1 and another t_2, with corresponding sediment heights z_1 and z_2. Use these and the definition of a linear operation to see if the relationship between t and z is linear.

$$r(t_1 + t_2 - t_0) = r(t_1 - t_0) + r(t_2 - t_0) + t_0 = z_1 + z_2 + t_0$$

The function is not linear, as the extra t_0 shows. It is affine.

2.2 Try again

Is the relationship between $t - t_0$ and z linear?

Yes.

2.3 Rates that vary

You could also consider the equation above as solution of a differential equation: $\frac{dz}{dt} = r$, with constant r. Consider a different differential equation $\frac{dz}{dt} = \frac{z}{\tau}$ with constant τ. Is this equation linear in z?

Yes, both the right and left hand sides are made of linear operators (derivative and multiplication by a scalar).

2.4 Solve the Diff Eq?

Does the solution to the differential equation $\frac{dz}{dt} = \frac{z}{\tau}$ with constant τ look like a straight line?

No. It is an exponential curve.

3 Complex Map: Earthquake Epicenter

An earthquake near San Francisco is detected at Eureka, Elko, and Las Vegas. A clever scientist decides to map the propagation of the quake waves on the surface using complex numbers, where N-S distance is the complex part of each number and E-W distance is the real part. (You can assume the Earth is flat over these short distances).

3.1 Distances

Suppose d_{01} is the (complex number a.k.a. vector) distance from the epicenter to Eureka, d_{01} is the (complex number a.k.a. vector) distance from the epicenter to Eureka, d_{12} is the distance from Eureka to Elko, etc. The distances between Eureka and Elko (d_{12}), Elko and Las Vegas (d_{23}) and Eureka and Las Vegas (d_{13}) are laid out in a triangle and related. What equation show this layout?

$$d_{12} + d_{23} = d_{13}$$

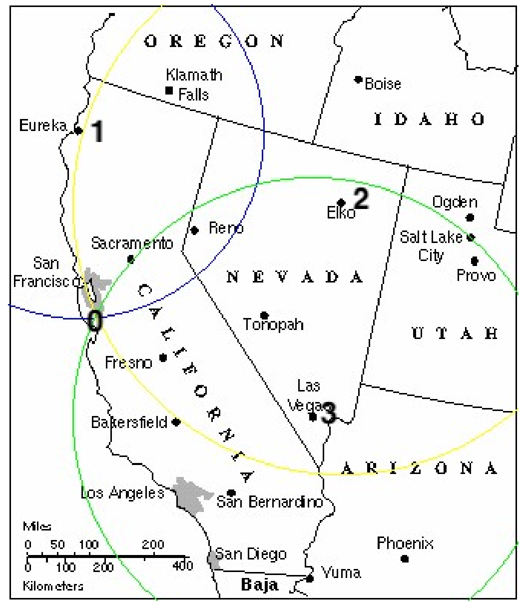

Figure 3: **Three sites–1) Eureka, 2) Elko, and 3) Las Vegas–detect an epicenter at 0) just south of San Francisco.**

3.2 3 More Triangles

Now consider 3 more triangles: Eureka to the Epicenter to Elko, Elko to Epicenter to Las Vegas, Eureka to Epicenter to Las Vegas. What equations are these?

$$-d_{01} + d_{02} = d_{12},$$
$$-d_{02} + d_{03} = d_{23},$$
$$-d_{01} + d_{03} = d_{13},$$

3.3 Solve

Using the equations for the 4 triangles above, and given d_{12}, d_{13}, d_{23}, can you solve this system for the location of the epicenter (i.e., find d_{01}, d_{02}, d_{03})? (Hint: write as a matrix equation and calculate the determinant)

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} d_{01} \\ d_{02} \\ d_{03} \end{bmatrix} = \begin{bmatrix} d_{12} \\ d_{23} \\ d_{13} \end{bmatrix}$$

The determinant is zero, so the system is not uniquely solvable

3.4 Speed to distance

Time t_{01} is the measured time the seismic waves took to travel from the epicenter to Eureka, t_{02} is the time from the epicenter to Elko, and t_{03} is the time from the epicenter to Las Vegas. If the speed of propagation, c, is the same in all directions and all locations (circles on the figure), write equations relating the distances $d_{01}, d_{02}, \&\ d_{03}$ to the scalar times t_{01}, t_{02}, t_{03}. With this additional information, can you calculate the epicenter?

$ct_{01} = \sqrt{d_{01}d_{01}^*}, ct_{02} = \sqrt{d_{02}d_{02}^*}, ct_{03} = \sqrt{d_{03}d_{03}^*}$. Yes, the additional constraints given by these times will eliminate the redundancy in the linear system. Interestingly, it is possible to be inconsistent–now the solution is overdetermined by the times and distances.

4 Preposterous!

Indicate why the following equations are "contrary to reason or common sense; utterly absurd or ridiculous" (Oxford English Dictionary).

4.1 Q1

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots \geq \tanh x \geq 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$$

LHS is convergent. RHS is divergent, but is less than convergent. Middle is bounded between -1 and 1.

4.2 Q2

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

LHS is a 3x3 matrix, RHS is a scalar (dot product).

4.3 Q3

$$\iiint_V \nabla\cdot(\nabla\times\mathbf{v})\,\mathrm{d}V = \oiint_A (\nabla\times\mathbf{v})\cdot\hat{\mathbf{n}}\,\mathrm{d}A = \oint_{\partial A} \mathbf{v}(\mathbf{l})\cdot\mathrm{d}\mathbf{l} = f(b)-f(a) = 1.$$

LHS is zero, because div of curl. Line integral is closed, so not $b-a$. Surface is closed, so not bounded by an outer edge.

4.4 Q4

I find the metric system so dull that I measure speed in seconds.

Dimensions! Speed is length per time. Also, seconds are part of the metric system.

4.5 Q5: Mid 16th century from Latin praeposterus 'reversed, absurd' (from prae 'before' + posterus 'coming after'). Modern: bass-akward

$$\mathbf{A}\times\nabla\times\mathbf{C} = \nabla(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\nabla), \qquad (\mathbf{A},\mathbf{C}) \text{ are vectors}$$

The BAC-CAB rule doesn't work with derivatives because it implies commutation of derivatives with vectors that are not allowed, also what is the ∇ acting on in the second term?