Fall 2021 EEPS0350–GeoMath Final Exam

1 To D'Alembert or not to D'Alembert

The nondispersive wave equation is a classic partial differential equation:

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \nabla^2 \phi. \tag{1}$$

Where ϕ is a property of the wave that propagates. In its simplest form of one spatial dimension x, the wave equation is

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2}. \tag{2}$$

1.1 Show that D'Alembert Solution Works

As we have shown in class and homeworks, show that $f(x-ct)$ and $g(x+ct)$ solve the one spatial dimension wave equation.

By application of the chain rule,

$$\frac{\partial^2 f(x-ct)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 f(x-ct)}{\partial t^2} = \frac{\mathrm{d}^2 f(x-ct)}{\mathrm{d}(x-ct)^2} - \frac{c^2}{c^2}\frac{\mathrm{d}^2 f(x-ct)}{\mathrm{d}(x-ct)^2} = 0.$$

$$\frac{\partial^2 g(x+ct)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 g(x+ct)}{\partial t^2} = \frac{\mathrm{d}^2 g(x+ct)}{\mathrm{d}(x+ct)^2} - \frac{c^2}{c^2}\frac{\mathrm{d}^2 g(x+ct)}{\mathrm{d}(x+ct)^2} = 0.$$

1.2 Separation of Variables

Use the method of separation of variables to separate out a time dependent part of the wave solution ($T(t)$) from a space dependent part ($X(x)$), so the assumed form is $\phi(t, x) = T(t)X(x)$ and the general solution is a sum over many of these terms. What equations do $X(x)$ and $T(t)$ solve?

Plugging into the wave equation, and dividing by TX,

$$\frac{\partial^2 T(t)X(x)}{\partial t^2} = X(x)\frac{\partial^2 T(t)}{\partial t^2} = c^2 T(t)\frac{\partial^2 X(x)}{\partial x^2} = c^2 \frac{\partial^2 T(t)X(x)}{\partial x^2} \tag{3}$$

$$\frac{1}{T(t)}\frac{\partial^2 T(t)}{\partial t^2} = \frac{c^2}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = \text{constant} \tag{4}$$

Thus, $X(x)$ and $T(t)$ are both equations where their second derivative is proportional to their value.

1.3 Linking

Based on our guess and check approach, show that the separation of variables approach leads to sines and cosines, i.e.,, plug in for $T(t) = \sin(\sigma t)$ and $X(x) = \sin(kx)$. (note that here σ and k are constants for your convenience).

Plugging into the curl equation, and dividing by TX,

$$\frac{1}{T(t)}\frac{\partial^2 T(t)}{\partial t^2} = \frac{-\sigma^2 \sin(\sigma t)}{\sin(\sigma t)} = -\sigma^2 \tag{5}$$

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = \frac{-k^2 \sin(kx)}{\sin(kx)} = -k^2 \tag{6}$$

Thus, $X(x)$ and $T(t)$ in this form solve the separation equations from the previous split.

1.4 Are these D'Alembert? Trig Identity...

These $T(t) = \sin(\sigma t)$ and $X(x) = \sin(kx)$ solutions are not of the D'Alembert form $f(x-ct)$ and $g(x+ct)$. Use the (Ptolemy) trig identifies to explain how one is related to the other if $c = \sigma/k$

$$\begin{aligned}
\sin\alpha\cos\beta &= [\sin(\alpha+\beta) + \sin(\alpha-\beta)]/2 \\
\cos\alpha\cos\beta &= [\cos(\alpha+\beta) + \cos(\alpha-\beta)]/2 \\
\sin\alpha\sin\beta &= [\cos(\alpha-\beta) - \cos(\alpha+\beta)]/2
\end{aligned}$$

Consider the product of $X(x) = \sin(kx)$ and $T(t) = \sin(\sigma t)$, this product is of the form of the last identity, thus

$$\begin{aligned}
\sin(kx)\sin(\sigma t) &= [\cos(kx - \sigma t) - \cos(kx + \sigma t)]/2 \\
&= [\cos(k(x - t\sigma/k)) - \cos(k(x + t\sigma/k))]/2 \\
&= [\cos(k(x - ct)) - \cos(k(x + ct))]/2
\end{aligned}$$

Thus, the sinusoid is a difference (linear combination) of two D'Alembert solutions to the wave equation. Since the wave equation is linear, this means this difference is also a solution to the wave equation.

2 The X Games

Consider the following three different probability distribution functions... particularly as they concern *extreme values*, taken here to be the values of x observed at greater than $\mu + 2\sigma$.

$$\rho_u(x;\sigma,\mu) = \frac{\mathcal{H}(\sqrt{3}\sigma - |x-\mu|)}{2\sqrt{3}\sigma}, \quad \rho_e(x;\sigma,\mu) = \frac{1}{\sigma\sqrt{2}}e^{-\sqrt{2}|x-\mu|/\sigma}, \quad \rho_n(x;\sigma,\mu) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)} \tag{7}$$

Where $\mathcal{H}$ is the Heaviside function (zero when its argument is negative and 1 when positive). The three distributions have the same mean μ and standard deviation σ. Half of the experimental data fall above the mean and half below.

Hints:

$$\int_{\mu}^{\infty} \rho_u(x;\sigma,\mu)\,\mathrm{d}x = 0.5, \quad \int_{\mu}^{\infty} \rho_e(x;\sigma,\mu)\,\mathrm{d}x = 0.5, \quad \int_{\mu}^{\infty} \rho_n(x;\sigma,\mu)\,\mathrm{d}x = 0.5 \tag{8}$$

$$\int_{\mu+\sigma}^{\infty} \rho_u(x;\sigma,\mu)\,\mathrm{d}x = 0.21, \quad \int_{\mu+\sigma}^{\infty} \rho_e(x;\sigma,\mu)\,\mathrm{d}x \approx 0.12, \quad \int_{\mu+\sigma}^{\infty} \rho_n(x;\sigma,\mu)\,\mathrm{d}x \approx 0.16, \tag{9}$$

$$\int_{\mu+2\sigma}^{\infty} \rho_u(x;\sigma,\mu)\,\mathrm{d}x = 0, \quad \int_{\mu+2\sigma}^{\infty} \rho_e(x;\sigma,\mu)\,\mathrm{d}x \approx 0.03, \quad \int_{\mu+2\sigma}^{\infty} \rho_n(x;\sigma,\mu)\,\mathrm{d}x \approx 0.02, \tag{10}$$

$$\int_{\mu+3\sigma}^{\infty} \rho_u(x;\sigma,\mu)\,\mathrm{d}x = 0, \quad \int_{\mu+3\sigma}^{\infty} \rho_e(x;\sigma,\mu)\,\mathrm{d}x \approx 0.007, \quad \int_{\mu+3\sigma}^{\infty} \rho_n(x;\sigma,\mu)\,\mathrm{d}x \approx 0.001. \tag{11}$$

2.1 How likely are extreme values?

Based on the defined probability distributions and their integrals, which of the three distributions is most likely to result in extreme values (those greater than $\mu + 2\sigma$) and which is the least likely?

The middle distribution or ρ_e has the greatest likelihood of extreme values because its integral from $\mu+2\sigma$ to ∞ is the largest. The ρ_u distribution has the least extreme values, as they are indeed of zero probability in that distribution.

2.2 Shifting the Mean

Suppose climate change affects the mean of the distributions (μ goes from $\mu_1 \to \mu_2$, $\mu_2 > \mu_1$, but μ_2 is only slightly larger than μ_1), but not the standard deviation ($\sigma = \sigma_1 = \sigma_2$). If the definition of extreme values stays fixed based on the pre-climate-change definition ($x > \mu_1 + 2\sigma_1$), are extreme values more likely before or after climate change sets in?

Since the change from μ_1 to μ_2 shifts the probability distribution toward larger values of x, this increases the likelihood of extreme values. The exception to this rule is the uniform distribution, which still has zero probability of extreme values so long as $\mu_2 - \mu_1 \leq (2-\sqrt{3})\sigma \approx 0.26795\sigma$

2.3 Taylor Series it Out

Use a Taylor series to estimate the changing rate of extreme values (where $x \geq \mu_1 + 2\sigma$) if the mean of the probability distribution shifts to a new μ near μ_1, for very small shifts in the mean. That is, form the leading order Taylor series near $\mu = \mu_1$ of the probability of finding an extreme value as a function of $\mu - \mu_1$.

Hints:

$$\left.\frac{\partial \rho_u(x;\sigma,\mu)}{\partial \mu}\right|_{x=\mu+2\sigma} = 0, \quad \left.\frac{\partial \rho_e(x;\sigma,\mu)}{\partial \mu}\right|_{x=\mu+2\sigma} \approx \frac{0.059}{\sigma^2}, \quad \left.\frac{\partial \rho_n(x;\sigma,\mu)}{\partial \mu}\right|_{x=\mu+2\sigma} \approx \frac{0.108}{\sigma^2},$$
$$\frac{\partial}{\partial \mu}\int_{\mu_1+2\sigma}^{\infty} \rho_u(x;\sigma,\mu)\,\mathrm{d}x = 0,$$
$$\left.\frac{\partial}{\partial \mu}\int_{\mu_1+2\sigma}^{\infty} \rho_e(x;\sigma,\mu)\,\mathrm{d}x\right|_{\mu=\mu_1} \approx \frac{0.042}{\sigma}, \quad \left.\frac{\partial}{\partial \mu}\int_{\mu_1+2\sigma}^{\infty} \rho_n(x;\sigma,\mu)\,\mathrm{d}x\right|_{\mu=\mu_1} \approx \frac{0.054}{\sigma}.$$

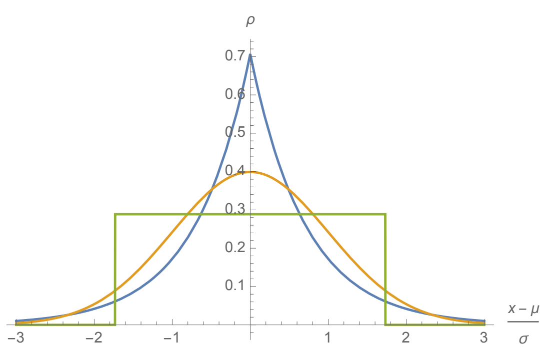
The desired Taylor series is formed from the integrals in (10)

$$\int_{\mu_1+2\sigma}^{\infty} \rho_u(x;\sigma,\mu)\,\mathrm{d}x = 0,$$
$$\int_{\mu_1+2\sigma}^{\infty} \rho_e(x;\sigma,\mu)\,\mathrm{d}x \approx 0.03 + \frac{0.042}{\sigma}(\mu-\mu_1) + \dots,$$
$$\int_{\mu+2\sigma}^{\infty} \rho_n(x;\sigma,\mu)\,\mathrm{d}x \approx 0.02 + \frac{0.054}{\sigma}(\mu-\mu_1) + \dots$$

The uniform distribution is eliminated by these results.

2.4 Graph em linear!

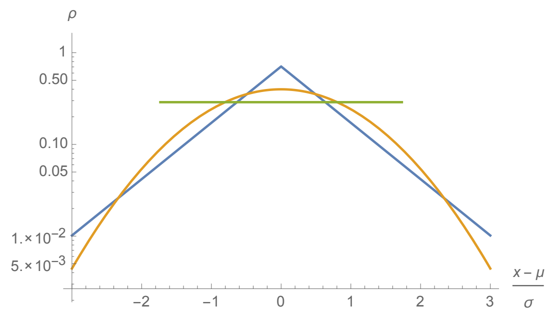
Here are the three probability distributions given between $-3\sigma + \mu_1$ and $+3\sigma + \mu_1$. Sketch their post-climate-change distributions on the same graph with $\mu_2 > \mu_1$ for a slightly larger μ_2 on linear axes. Shade the area indicating a change in the extreme value probability.



The graph should just show the distributions shifted slightly to the right. Everywhere from $\mu + 2\sigma$ to the right in the area between the old curve and the new shifted pdf should be shaded in.

2.5 Graph em log!

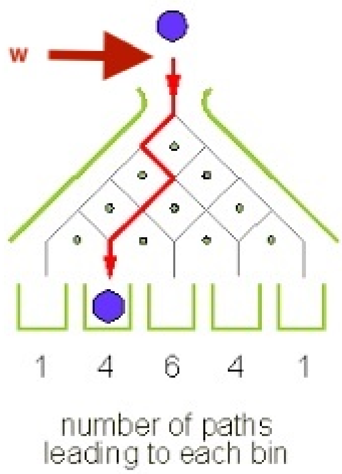
Here are the three probability distributions given between $-3\sigma + \mu_1$ and $+3\sigma + \mu_1$. Sketch their post-climate-change distributions on the same graph with $\mu_2 > \mu_1$ for a slightly larger μ_2 on semi-log axes. Shade the area indicating a change in the extreme value probability.



The graph should just show the distributions shifted slightly to the right. Everywhere from $\mu + 2\sigma$ to the right in the area between the old curve and the new shifted pdf should be shaded in.

3 Less than Random

A physical system to make a binomial distribution drops a ball through a series of pins. At each pin, the ball can either bounce to the right or to the left of the pin.



Consider using such a system in a windy place (see wind direction arrow and wind strength w). We might write a dynamical system for the ball's behavior upon reaching the first pin as:

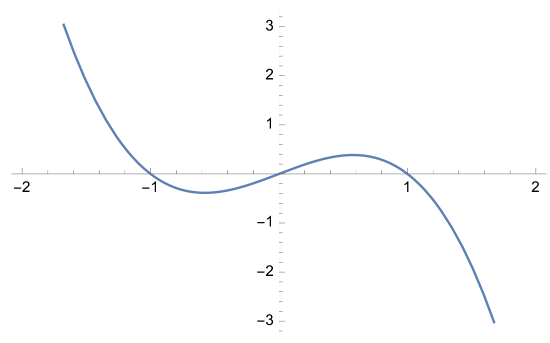
$$\dot{x} = w - zx - x^3. \tag{12}$$

where x is the location of the ball in the horizontal as it encounters the first pin, z is its location in the vertical ($z = 0$ is the location of the pin, $z > 0$ is above the pin, and $z < 0$ is below the pin), and w is the wind strength.

3.1 No Wind!

Plot a phase diagram for the case where $w = 0, z = -1$ over the range $-2 < x < 2$. Draw a histogram/probability distribution function of the likely outcomes (left ($x < 0$) vs. right ($x > 0$) of the pin). Put arrows to indicate the direction of rate of change and use circles to denote stable and unstable fixed points.

We draw the function, noting their relative values and shapes. The histogram would just be two equal bars, one for left and one for right.



3.2 Windy!

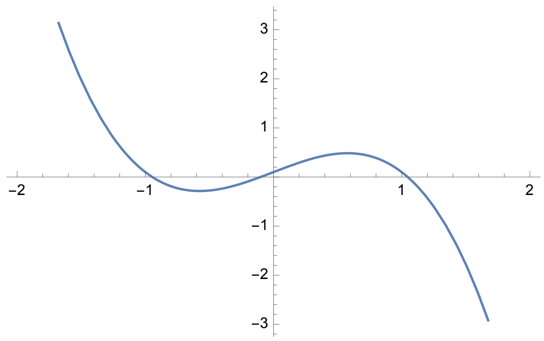
Plot a phase diagram for the case where $w = 0.1, z = -1$ over the range $-2 < x < 2$. Draw a histogram/probability distribution function of the likely outcomes (left ($x < 0$) vs. right ($x > 0$) of the pin). Put arrows to indicate the direction of rate of change and use circles to denote stable and unstable fixed points.

We draw the function, noting their relative values and shapes. The histogram would just be two unequal bars, the one for left being slightly smaller and one for right being slightly larger.

3.3 Bifurcation!

Plot a bifurcation diagram over the range $-1 < z < 1$ with $w = 0.1$ (or really any small number).

We draw the function, noting their relative values and shapes.


3
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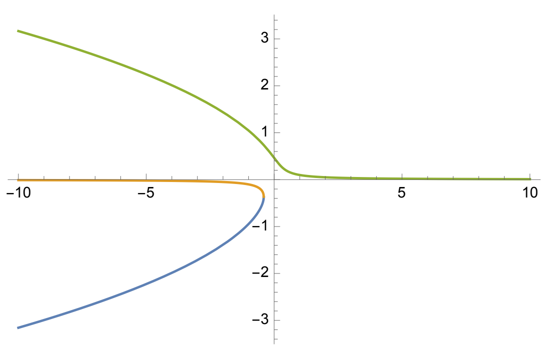
3
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4 Curly Stress

The momentum equation for a solid, liquid, or gas is.

$$\rho\frac{D\mathbf{v}}{Dt} = \nabla \cdot \sigma + \rho\mathbf{g}. \tag{13}$$

Consider the specific stress tensor

$$\sigma = \begin{bmatrix} w_y - v_z & w_y - v_z & w_y - v_z \\ u_z - w_x & u_z - w_x & u_z - w_x \\ v_x - u_y & v_x - u_y & v_x - u_y \end{bmatrix} \tag{14}$$

Where (u, v, w) are the three components of $\mathbf{v}$, and subscripts denote partial derivatives, e.g., $w_y = \frac{\partial w}{\partial y}$.

Hint: with this notation, note that $\nabla \times \mathbf{v} = (w_y - v_z)\hat{\mathbf{i}} + (u_z - w_x)\hat{\mathbf{j}} + (v_x - u_y)\hat{\mathbf{k}}$.

4.1 Symmetric Stress Tensor?

Is the specified stress tensor symmetric? Is this an issue? Why?

No, the stress tensor is not symmetric. Stress tensors must be symmetric in order to conserve angular momentum (from the notes and discussion).

4.2 Divergence of this Stress

Calculate the divergence of this stress tensor.

From the hint, we note that each column of the stress tensor is actually a curl. Thus, the divergence of the stress tensor, which is just the divergence of each column three times, is zero. This could also be worked out one step at a time by distributing out the divergence operator on the columns of the tensor.

4.3 Meaning?

With the behaviors above about the symmetry and divergence of this stress tensor, explain what the physical meaning of this particular stress tensor is.

In this case, $0 = \nabla \cdot \sigma$, so there are no stresses implied for any velocities. Because there are no contributions from the stress tensor, it is not a problem that the tensor is not symmetric, as it plays no role in the dynamics. Thus, concerns that the stress tensor is antisymmetric is not a problem ;-).