1.11 Homework Problems

1.11.1 Manipulation

Exercise 1.1

$$1 + 0.1 + 0.01 + 0.001 + \dots = 1.1111\dots \tag{1.53}$$

Which can be understood by the sequence of partial sums S_n, or from (1.4) with $a = 1$ and $r = 0.1$.

Exercise 1.2

$$1 + 1/2 + 1/4 + 1/8 + \dots = 2 \tag{1.54}$$

Which can be understood by the residual of the sequence of partial sums versus 2, $2 - S_n = 1, 1/2, 1/4, 1/8, \dots$, or from (1.4) with $a = 1$ and $r = 1/2$.

Exercise 1.3

$$1 + 1 + 1 + 1 + \dots = \infty \tag{1.55}$$

Which can be understood by the sequence of partial sums being the natural numbers.

Exercise 1.4 (Taylor Realize) *Confirm the first two nonzero terms of each of the series in (1.24), (1.26), and (1.28) by plugging into those formulae for $n = 0, 1, 2, \dots$. For some n, the terms will vanish, so keep going until you get two nonzero ones. Then, use (1.16) and derive the first two nonzero terms in the series by taking the derivatives of each function. Again, some terms will vanish (and they may not match up one-to-one with the formula in (1.23-1.28) until you have enough terms calculated).*

(For clarity, three terms are worked out here) From (1.23-1.28)

$$\cos x = \frac{(-1)^0 x^0}{(0)!} + \frac{(-1)^1 x^2}{(2)!} + \frac{(-1)^2 x^4}{(4)!} + \dots, \tag{1.56}$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24} + \dots. \tag{1.57}$$

$$\ln(1+x) = \frac{(-1)^2 x^1}{1} + \frac{(-1)^3 x^2}{2} + \frac{(-1)^4 x^3}{3} + \dots, \tag{1.58}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} + \dots. \tag{1.59}$$

$$(1+x)^p = \binom{p}{0} x^0 + \binom{p}{1} x^1 + \binom{p}{2} x^2 + \binom{p}{3} x^3 + \dots, \tag{1.60}$$

$$= 1 + \frac{p!}{1!(p-1)!} x + \frac{p!}{2!(p-2)!} x^2 + \frac{p!}{3!(p-3)!} x^3 + \dots, \tag{1.61}$$

$$= 1 + \frac{p!}{(p-1)!} x + \frac{p!}{2(p-2)!} x^2 + \frac{p!}{6(p-3)!} x^3 + \dots, \tag{1.62}$$

$$= 1 + px + \frac{1}{2} p(p-1) x^2 + \frac{1}{6} p(p-1)(p-2) x^3 + \dots. \tag{1.63}$$

From (1.16)

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-0)^n}{n!} f^{(n)}(0) = f(0) + xf'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{6} f'''(0) + \dots \tag{1.64}$$

$$\cos(x) = \cos(0) + x\cos'(0) + \frac{x^2}{2}\cos''(0) + \frac{x^3}{6}\cos'''(0) + \dots, \tag{1.65}$$

$$= 1 - x\sin(0) - \frac{x^2}{2}\cos(0) + \frac{x^3}{6}\sin(0) + \dots, \tag{1.66}$$

$$= 1 - \frac{x^2}{2} + \dots. \tag{1.67}$$

$$\ln(1+x) = [\ln(1+x)] + x[\ln(1+x)]'(0) + \frac{x^2}{2}[\ln(1+x)]''(0) + \frac{x^3}{6}[\ln(1+x)]'''(0) + \dots, \tag{1.68}$$

$$= 0 + x\frac{1}{1+0} + \frac{x^2}{2}\frac{-1}{(1+0)^2} + \frac{x^3}{6}\frac{2}{(1+0)^3} + \dots, \tag{1.69}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} + \dots. \tag{1.70}$$

$$(1+x)^p = (1+0)^p + x[(1+x)^p]'(0) + \frac{x^2}{2}[(1+x)^p]''(0) + \frac{x^3}{6}[(1+x)^p]'''(0) + \dots, \tag{1.71}$$

$$= 1 + px(1+0)^{p-1} + p(p-1)\frac{x^2}{2}(1+0)^{p-2} + p(p-1)(p-2)\frac{x^3}{6}(1+0)^{p-3} + \dots, \tag{1.72}$$

$$= 1 + px + p(p-1)\frac{x^2}{2} + p(p-1)(p-2)\frac{x^3}{6} + \dots. \tag{1.73}$$

1.11.2 Application

Exercise 1.5 (30N) *An important parameter in the consideration of the physics of the rotating Earth is the Coriolis parameter: $2\Omega\sin(\varphi)$, where Ω is the angular rate of rotation of the earth in radians (2π in a day, or $2\pi/(24hr/s\cdot 3600s)$) and φ is the latitude. Taylor expand the first terms in this parameter around 30 degrees North, or $30^\circ\cdot(2\pi radians/360^\circ) = \pi/6$ radians. Carry out the expansion to the term including $(\varphi-\pi/6)^3$.*

$$\sin(\varphi) = \sum_{n=0}^{\infty} \frac{(\varphi-\frac{\pi}{6})^n}{n!}\sin(\varphi)^{(n)}\left(\frac{\pi}{6}\right), \tag{1.74}$$

$$= \frac{(\varphi-\frac{\pi}{6})^0}{0!}\sin(\varphi)^{(0)}\left(\frac{\pi}{6}\right) + \frac{(\varphi-\frac{\pi}{6})^1}{1!}\sin(\varphi)^{(1)}\left(\frac{\pi}{6}\right) + \frac{(\varphi-\frac{\pi}{6})^2}{2!}\sin(\varphi)^{(2)}\left(\frac{\pi}{6}\right) + \dots,$$

$$= \sin\left(\frac{\pi}{6}\right) + (\varphi-\frac{\pi}{6})\cos\left(\frac{\pi}{6}\right) - \frac{(\varphi-\frac{\pi}{6})^2}{2}\sin\left(\frac{\pi}{6}\right) - \frac{(\varphi-\frac{\pi}{6})^3}{6}\cos\left(\frac{\pi}{6}\right) + \dots, \tag{1.75}$$

$$= \frac{1}{2} + (\varphi-\frac{\pi}{6})\frac{\sqrt{3}}{2} - \frac{(\varphi-\frac{\pi}{6})^2}{4} - \frac{(\varphi-\frac{\pi}{6})^3\sqrt{3}}{12} + \dots, \tag{1.76}$$

$$2\Omega\sin(\varphi) = \Omega + \Omega(\varphi-\frac{\pi}{6})\sqrt{3} - \Omega\frac{(\varphi-\frac{\pi}{6})^2}{2} - \Omega\frac{(\varphi-\frac{\pi}{6})^3\sqrt{3}}{6} + \dots \tag{1.77}$$

Exercise 1.6 (Rocking with Taylor Swiftly) *Consider pushing on the rock depicted in Fig. 3.2. If you push or pull gently, the rock will push back. If you push or pull hard, not so much. We will*

Figure 1.4: **A situation that's sensitive to perturbation amplitude.**

use this example to consider how nonlinear functions are sensitive to amplitude in a way that linear functions are not.

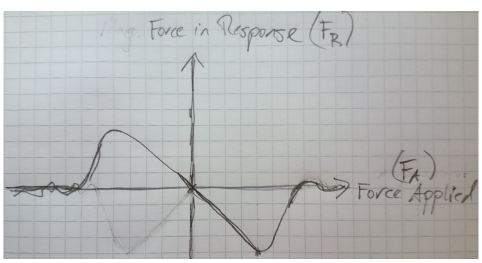
a, Equal and Opposite: *Make a graph depicting the force applied, F_A (positive=push in horizontal direction, negative=pull in horizontal direction) versus the force back from the rock F_R (after equilibration). You can suppose that all forces go in the same direction. (Hint: prevent acceleration up to a point, and $F_A + F_R = ma$.*

b, Constant Approximation: *If the function $F_R(F_A)$ is fit with a Taylor series around $F_A = 0$ and truncated at the first (constant) term ($F_R(F_A) \approx c_0$). Describe this system's response to applied forces.*

c, Linear Approximation: *If the Taylor series for $F_R(F_A)$ around $F_A = 0$ is truncated after two terms ($F_R \approx c_0 + c_1 F_A$), it predicts how much F_R for $F_A = 1$ Newton? 2 Newtons? 10^5 Newtons? Can fracture happen?*

d, Breaking Point: *What is the minimum number of terms in the Taylor series that must be retained to make F_R do something other than double when F_A is doubled?*

e, Extremes: *Consider the response force F_R at positive and negative F_A. Is it approximately an even or odd function? If this were exactly true (i.e., symmetry or antisymmetry between pushing and pulling) what is the first term in the Taylor series that could make F_R sensitive to amplitude of F_A?*

a: 

b: As $F_R(0) = c_0 = 0$, this system has no response, so $ma = F_A$.

c: $F_R = -1$Newton, -2Newtons, -10^5Newtons. This system cannot fracture, as $F_R = -F_A$.

d: At least one nonlinear term must be retained.

e: It is an odd function. Thus, we expect the $c_3(F_A)^3$ term to be the first amplitude sensitive one.

Exercise 1.7 (Taylor rhymes with Baylor) *Suppose the function $h(x)$ plotted in the figure is found by measuring topography along the x direction, and in particular consider fitting it with a* *series expansion near the point marked A. Sea level is $h = 0$, so we are particularly interested in $h(x) = 0$, which indicates the location of coastlines. The function is not known, but we consider the possibility of approximating a Taylor series expansion to it.*

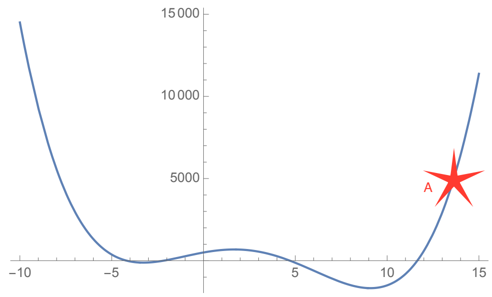


Figure 1.5: A function to be fit by Taylor expansion near point A at the star.

a, Counting: *How many coastlines are there, that is how many solutions to $h(x) = 0$ are there?*
b, Constant: *If the function is fit with a Taylor series and truncated at the first (constant) term, and if the truncated approximation is denoted $\tilde{h}_0(x)$, then how many solutions are there to $\tilde{h}_0(x) = 0$?*
c, Variations: *If the Taylor series is instead truncated after two terms ($\tilde{h}_1(x)$) then how many solutions are there to $\tilde{h}_1(x) = 0$, and what is the approximate value of the solution x?*
d, How many?: *What is the minimum number of terms in the Taylor series that must be retained to approximate all of the coastlines in the real function $h(x)$? Why?*
e, Extremes: *Consider the function at large magnitudes of positive and negative x. If the truncated Taylor series matches this behavior at large $|x|$, predict the sign of the coefficient in the largest power of x and whether the power is even or odd.*

a: 4

b: None.

c: One, near $x = 12$.

d: Since there are 4 roots to the equation $h(x) = 0$, $\tilde{h}(x)$ must be at least a fourth-order polynomial to have this many roots, which has 5 terms of the Taylor series retained.

e: Positive and even power. Negative would tend toward $-\infty$, and even power since the function increases on both sides of A or 0.

1.11.3 Scheming Schematics and Articulate Analysis

Exercise 1.8 (Who is the Oddest?) *Sometimes it is said that* sin *is an* odd function *and* cos *is an* even function. *Examine the* sin *and* cos *functions in (1.23) and explain what this means in terms of the exponents of x. Consider $\sin(x)$ versus $\sin(-x)$ and $\cos(x)$ versus $\cos(-x)$, how do they compare? How do odd and even functions compare under the sign reversal of their argument (i.e., the input to the function, x or $-x$) in general?*

The powers of all of the terms in the (1.23) are either all odd (in the case of sine) or all even (in the case of cosine). As odd powers reverse sign under reversal of the argument, e.g., $(-x)^3 = -x^3$, $\sin(-x) = -\sin(x)$. By contrast e.g., $(-x)^4 = x^4$, so $\cos(-x) = \cos(x)$. Also, since taking the derivative lowers the exponent of each term by one, it makes sense that the derivative of an odd function is even and vice versa.

Exercise 1.9 (Biggie Smalls) *Examine each factor in the product that makes up (1.16). What makes them large or small as n increases?*

It is important to think a bit about what leads the Taylor series to converge. The $n!$ in the denominator is a strong pull toward smaller and smaller numbers. In the examples (1.23)-(1.29), this factorial remains in many of the series. However, keeping $(x-a)$ small is also important–that is, the approximation is better the closer to the location where the derivatives are evaluated. Finally, the derivatives themselves need to not get larger and larger with repeated differentiation. In series where this occurs, such as (1.29), the factorial is cancelled out by the increasingly sharp derivative functions.
The factorial always decreases the size with increasing n. The $(x-a)^n$ can increase or decrease the size, depending on whether x is close to a. The derivative $f^{(n)}(a)$ tends to be noisier & larger with increasing n (as integrals tend to be smoother than the original signal, hence averaging). Examining the finite difference approximation for the following combination,

$$\frac{(x-a)^n \Delta^n f}{\Delta x^n}, \tag{1.78}$$

implies that "near" and "far" for $(x-a)$ is measured in terms of how far in x you need to go to make Δf sizeable or make f "wiggle" appreciably. The units of f provide the units of the whole Taylor approximation, while the units of $x-a$ need to match the units of Δx.

Exercise 1.10 (Getting Seri-us) *The Taylor series and the Fourier series are methods to determine the coefficients in series expansions of a function $f(x)$. They apply in approximating the function near $x = a$ and over a given interval where the function is periodic. Suppose we have a function that is periodic when $-\pi \le x - a \le \pi$, then we can write the Taylor and Fourier series–for the* same function *$f(x)$ as,*

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots,$$
$$f(x) = d_0 + d_1 e^{i(x-a)} + d_2 e^{2i(x-a)} + d_3 e^{3i(x-a)} + \dots$$

Use these forms to answer the following questions. Hint:

$$e^{ni(x-a)} = 1 + in(x-a) - \frac{n^2}{2}(x-a)^2 - \frac{in^3}{6}(x-a)^3 + \dots + \frac{(in)^m}{m!}(x-a)^m + \dots$$

a, Constant: *Is there a simple relationship between c_0 and d_0 very near $x = a$ for all $f(x)$?*
b, Variations: *Does a specific term in the Fourier series, say $d_2 e^{2i(x-a)}$ always correspond to a specific term in the Taylor series, regardless of what $f(x)$ is?*
c, Equivalence?: *Assuming both series converge, is there a unique (though perhaps complicated) relationship among all of the c_n and all of the d_n?*
d, Specifically: *If $f(x) = e^{i(x-a)}$, what are the values of all nonzero d_n and c_n?*

a: No. Setting $x = a$ yields $c_0 = d_0 + d_1 + d_2 + d_3 + \dots$

b: No. Every term in the Fourier series will contribute to every term in the Taylor series, since each has a part of it that goes as $(x-a)$ to any given power, as seen clearly in the hint.

c: Yes. There is a unique Taylor series and a unique Fourier series for a given function $f(x)$, and either can be used to derive the other, so there is a unique relationship among all of the coefficients of each.

d: $d_1 = 1$, all other d_n are zero. c_n are the coefficients given in the hint, $c_m = \frac{(i)^m}{m!}$.

1.11.4 Jargon to Argot

Exercise 1.11 (Sequence vs. Series) *What's the difference between a sequence and a series?*

A sequence is an ordered list of numbers, while a series is a *sum* of a sequence of numbers.

Exercise 1.12 (Asymptote, Asymptotic, Limit) *An asymptote is a line that continually approaches a given curve but does not meet it at any finite distance. Contrast this definition to that of the adjective* asymptotic *and the term* limit*.*

An asymptote is a line that continually approaches a curve. A behavior is asymptotic if it becomes increasingly true as a particular limit is approached, but it is not true at any finite distance. A limit is a value that is approached by a function as a parameter becomes large or small. The asymptote is a graphical description of an asymptotic limit.