HW on Chps 2&3 Answer Key:

Exercise 2.2 (Crunch your abs!) *Similar to problems in 2.5 of Boas (2006). Simplify the following number, z, to the $z = x + iy$ form and to the $z = re^{i\theta}$ form. Then plot the number and all of its fourth roots in the complex plane (Boas, 2006, see Section 2.10).*

$$z = \left[\frac{1+i}{1-i}\right]^4 \tag{2.16}$$

$$z = \left[\frac{1+i}{1-i}\right]^4 = \left[\frac{(1+i)(1-i)}{(1-i)(1-i)}\right]^4 = \left[\frac{2}{-2i}\right]^4 = i^4 = 1$$

Exercise 2.5 (The Buoyancy, or Brunt-Väisälä, Frequency) *In a density stratified fluid, displacing a fluid parcel (without changing its density) upward or downward results in a restoring buoyancy force, because a parcel displaced upward will be denser than its neighbors and a parcel displaced downward will be more buoyant than its neighbors. The equation that describes the motion for the position of the parcel Z can be written*

$$\frac{d^2 Z}{dt^2} = -N^2 Z. \tag{2.17}$$

Where N is a function of the density (ρ) stratification in the vertical direction (z) as compared to a background density ρ_0 and gravitational acceleration g:

$$N^2 = \frac{-g}{\rho_0}\frac{d\rho}{dz}. \tag{2.18}$$

The frequency N is called the buoyancy frequency or Brunt-Väisälä frequency after David Brunt and Vilho Väisälä. Verify that $Z = e^{iNt}, Z = e^{-iNt}, Z = \cos(Nt)$, and $Z = \sin(Nt)$ satisfy this equation.

$$\begin{aligned}
2.16.13: \frac{d^2}{dt^2}e^{iNt} &= i^2N^2e^{iNt} = -N^2(e^{iNt}), \\
\frac{d^2}{dt^2}e^{-iNt} &= (-i)^2N^2e^{-iNt} = -N^2(e^{-iNt}), \\
\frac{d^2}{dt^2}\sin(Nt) &= -N^2\sin(Nt) = -N^2(\sin(Nt)), \\
\frac{d^2}{dt^2}\cos(Nt) &= -N^2\cos(Nt) = -N^2(\cos(Nt)).
\end{aligned}$$

3)

Solve the following equations for unknowns (x, y, z), using row reduction (adding together multiples of equations to isolate variables, then back-substitute).

a)

$$x - 2y + 13 = 0,$$
$$y - 4x = 17$$

Add 2 times the second equation to the first,

$$x - 2y + 13 = 0,$$
$$-8x + 2y - 34 = 0$$

To find

$$-7x - 21 = 0,$$
$$-7x = 21,$$
$$x = -3.$$

Plug back into one of the original equations and

$$y = 4x + 17 = -12 + 17 = 5$$

Confirm by plugging into the other original equation.

$$x - 2y + 13 = -3 - 10 + 13 = 0$$

Confirmed.

b)

$$x - 2y = 4,$$
$$5x + z = 7,$$
$$x + 2y - z = 3$$

Simplify the first equation for x, use it to eliminate x in the other equations.

$$x = 2y + 4,$$
$$5x + z = 10y + 20 + z = 7,$$
$$x + 2y - z = 3 = 4y + 4 - z.$$

Simplify

$$\begin{aligned} 10y + z &= -13, \\ 4y - z &= -1. \end{aligned}$$

Add the last two equations to eliminate z,

$$\begin{aligned} 14y &= -14, \\ y &= -1. \end{aligned}$$

Backsubstitute

$$\begin{aligned} x &= -2 + 4 = 2 \\ x + 2y - z = 2 - 2 - z &= 3, \\ z &= -3. \end{aligned}$$

c)

$$\begin{bmatrix} 1 & -2 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -13 \\ 17 \end{bmatrix},$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{-1}{7} \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} -13 \\ 17 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 5 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 3 \end{bmatrix},$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1/2 & 1/2 \\ -5 & 2 & -5 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$$

4)

$$M = \begin{bmatrix} 5 & -2 \\ 2 & 0 \end{bmatrix}$$

a)

$$q = \sqrt{25 - 16} = 3$$
$$v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \lambda_1 = 4$$
$$v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \lambda_2 = 1$$

b)

$$\begin{bmatrix} 5 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 5 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

c)

$$\begin{bmatrix} 5 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

3.13.2 Application

Exercise 3.5 (Nontrivial Cramer's) *In Pedlosky (1987), Cramer's rule is repeatedly used to determine the dispersion relation for waves and instabilities that solve complex linear systems of equations. One example is the derivation of Kelvin and Poincaré waves in a channel (x is along-channel distance and y is the cross-channel distance, and L is the channel width). The waves (in displacement of the ocean surface, or η) are assumed to have the form*

$$\eta = Re\,(A\cos\alpha y + B\sin\alpha y)\,e^{i(kx-\sigma t)} \tag{3.63}$$

The parameter k is the wavenumber in x, σ is the frequency, and α is the wavenumber in y. A and B are amplitudes. In the derivation, the wave equations were used to show that must α depend on other parameters (C_0, a typical wave speed and f, the Coriolis parameter) in the following way: $\alpha^2 = \frac{\sigma^2-f^2}{C_0^2} - k^2$. The remaining equations (the boundary conditions at the walls of the channel) were boiled down to the following linear equations on A and B.

$$\alpha A \qquad + \frac{fk}{\sigma}B = 0, \tag{3.64}$$

$$\left[\alpha\cos\alpha L + \frac{fk}{\sigma}\sin\alpha L\right]A \qquad + \left[\frac{fk}{\sigma}\cos\alpha L - \alpha\sin\alpha L\right]B = 0. \tag{3.65}$$

Using Cramer's rule, prove that: a) If the determinant of the coefficients of A and B doesn't vanish, then the only solution is $A = 0$, $B = 0$. b) That a nontrivial solution is possible if the determinant vanishes, and show that a vanishing determinant is equivalent to the condition (called the dispersion relation which is used to solve for frequency given wavenumber or vice versa):

$$(\sigma^2 - f^2)(\sigma^2 - C_0^2k^2)\sin\alpha L = 0. \tag{3.66}$$

Finally, c) the equations for A and B are linear, but the dispersion relation between σ and k is not. Which operation in the use of Cramer's rule will virtually guarantee nonlinear polynomials? (Hint: the order of the polynomials will be closely related to the number of columns or rows in the coefficient matrix)

a) This is a homogeneous set of equations, so Cramer's rule tells us the solution will be trivial ($A = 0, B = 0$) because the column vector on the RHS is equal to zero, thus the numerator (determinant of a matrix with a column replaced by the RHS vector) in Cramer's rule will always

be equal to zero. The only way for a nontrivial solution in this case is if the denominator (the coefficient matrix's determinant) is zero.
b) We want a nontrivial solution, so set the determinant to be zero.

$$\alpha\left[\frac{fk}{\sigma}\cos\alpha L - \alpha\sin\alpha L\right] - \frac{fk}{\sigma}\left[\alpha\cos\alpha L + \frac{fk}{\sigma}\sin\alpha L\right] = 0,$$

$$\left(-\alpha^2 - \frac{f^2k^2}{\sigma^2}\right)\sin\alpha L = \left(-\frac{\sigma^2 - f^2}{C_0^2} + k^2 - \frac{f^2k^2}{\sigma^2}\right)\sin\alpha L = 0,$$

$$(\sigma^2 - f^2)(\sigma^2 - C_0^2k^2)\sin\alpha L = 0.$$

c) Finding the determinant of the coefficient matrix involves multiplying coefficients. Thus, the dispersion relation is usually not a linear function, as it is built from products of all of the coefficients. In this case, there are 3 classes of roots (solutions, or types of waves), $\sigma = \pm f$ (inertial oscillations), $\sigma = \pm C_0 k$ (gravity waves), and $\alpha L = n\pi$ (Poincaré Waves).