4.3.2 Manipulation

Exercise 4.2 (The Dims are Out!) *Show that Ro, Re, and Ra are dimensionless by expanding their parameters into their dimensions.*

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$$
\begin{aligned}
[\mathrm{Ro}] &= \left[\frac{U}{fL}\right] = \frac{\frac{L}{T}}{\frac{1}{T}L} = 1,\\
[\mathrm{Re}] &= \left[\frac{UL}{\nu}\right] = \frac{\frac{L}{T}L}{\frac{L^2}{T}} = 1,\\
[\mathrm{Ra}] &= [\mathrm{Gr}]\,[\mathrm{Pr}] = \left[\frac{g\Delta\rho L^3}{\nu^2\rho_0}\right]\left[\frac{\nu}{\kappa}\right] = \frac{\frac{L}{T^2}\frac{M}{L^3}L^3}{\frac{L^4}{T^2}\frac{M}{L^3}}\frac{\frac{L^2}{T}}{\frac{L^2}{T}} = 1\times 1.
\end{aligned}
$$

4.3.3 Application

Exercise 4.3 *(From 2.8 of* **?***)*
The shape of a hanging drop of liquid satisfies the following empirical equation

$$\frac{(\rho - \rho_a) g d^3}{C} = \sigma \tag{4.18}$$

Where ρ, ρ_a are the densities of the drop and air, g is gravitational acceleration, d is drop diameter, σ is surface tension (units of Newtons per meter) and C is an empirical constant. What are the units of C?

$$\frac{\left[(\rho - \rho_a) g d^3\right]}{[C]} = [\sigma] = \frac{\frac{M}{L^3}\frac{L}{T^2}L^3}{[C]} = \left[\frac{N}{m}\right] = \frac{ML}{T^2 L},$$
$$[C] = \frac{\frac{M}{L^3}\frac{L}{T^2}L^3}{\frac{ML}{T^2 L}} = L.$$

Exercise 4.4 (Run It Up the Flagpole) *(From 2.36 of* **?***)*
Because of a phenomenon called vortex shedding, a flagpole will oscillate at a frequency ω when the wind blows at velocity U. The diameter of the flagpole is D and the kinematic viscosity of air is ν. Using dimensional analysis, develop an equation for ω as the product of a quantity independent of ν with the dimensions of ω and a function of all relevant dimensionless groupings.

We have the following dimensional parameters and results,

$$[\omega] = \frac{1}{T}, \qquad [U] = \frac{L}{T}, \qquad [D] = L, \qquad [\nu] = \frac{L^2}{T},$$

Thus, Buckingham's Pi Theorem predicts $4 - 2 = 2$ independent dimensionless parameters. We choose the first as the Reynolds number of the pole. The second needs to incorporate the frequency, and we see that the Strouhal number works well for this parameter.

$$\mathrm{Re} = \frac{UD}{\nu}, \qquad \mathrm{St} = \frac{\omega D}{U}$$

Other choices would be acceptable–in particular any power of these two ratios! If there is only one dimensionless parameter, it must be constant. If there are two, then we propose that a function relates them. If we make this assumption, and then solve for the frequency,

$$\frac{\omega D}{U} = \mathrm{St} = f(\mathrm{Re}) = f\left(\frac{UD}{\nu}\right) \longrightarrow \omega = \frac{U}{D} f(\mathrm{Re}).$$

Quod erat inveniendum.

5.7.2 Application

Exercise 5.6 (Streamfunction and Velocity Potential) *As we come to understand how fluids move, we will be interested in breaking up the velocity field into a part that results from a convergence or divergence of flow and a part that results from a swirling or rotating flow. This can be done with the Helmholtz Decomposition, which we will address in a later chapter on vector analysis. For now, we'll just test some examples.*

The relationship between velocity (u, v) and streamfunction (ψ) and velocity potential (ϕ), in 2D, is just

$$u = -\frac{\partial \psi}{\partial y} - \frac{\partial \phi}{\partial x}, \tag{5.10}$$

$$v = \frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}. \tag{5.11}$$

Both velocity components, as well as ψ and ϕ should be interpreted as spatial fields–that is, they are all functions of x and y and have a value at every point in space.

The Jacobian is a useful function for evaluating the advection by flow due to a streamfunction alone, which in 2D is just

$$J(A, B) = \begin{vmatrix} \frac{\partial A}{\partial x} & \frac{\partial A}{\partial y} \\ \frac{\partial B}{\partial x} & \frac{\partial B}{\partial y} \end{vmatrix} = \frac{\partial A}{\partial x}\frac{\partial B}{\partial y} - \frac{\partial A}{\partial y}\frac{\partial B}{\partial x}. \tag{5.12}$$

a) If $\phi = 0$, show that $J(\psi, f(x, y)) = u\,\frac{\partial f(x, y)}{\partial x} + v\,\frac{\partial f(x, y)}{\partial y}$.
b) If A is a function of B, rather than independently varying in x and y, show that $J(B, A(B)) = 0$.

$$(a) J(\psi, f(x, y)) = \frac{\partial \psi}{\partial x}\frac{\partial f(x, y)}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial f(x, y)}{\partial x} = u\,\frac{\partial f(x, y)}{\partial x} + v\,\frac{\partial f(x, y)}{\partial y}.$$

$$(b)\text{Chain rule: } J(B, A(B)) = \frac{\partial B}{\partial x}\frac{\partial A(B)}{\partial y} - \frac{\partial B}{\partial y}\frac{\partial A(B)}{\partial x} = \frac{\partial B}{\partial x}\frac{\mathrm{d}A(B)}{\mathrm{d}B}\frac{\partial B}{\partial y} - \frac{\partial B}{\partial y}\frac{\mathrm{d}A(B)}{\mathrm{d}B}\frac{\partial B}{\partial x} = 0.$$

5.7.3 Evaluate & Create

Exercise 5.7 (The D'Alembert Wave) *Problem 4.7.23 of* **?**.
If $u = f(x - ct) + g(x + ct)$*, show that* $\dfrac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2}$.

We can do this by direct integration, but it is more fun to note that both $f(x - ct)$ and $g(x + ct)$ solve the wave equation (they are D'Alembert's solutions),

$$\frac{\partial^2 f(x-ct)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 f(x-ct)}{\partial t^2} = \frac{\mathrm{d}^2 f(x-ct)}{\mathrm{d}(x-ct)^2} - \frac{c^2}{c^2}\frac{\mathrm{d}^2 f(x-ct)}{\mathrm{d}(x-ct)^2} = 0.$$
$$\frac{\partial^2 g(x+ct)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 g(x+ct)}{\partial t^2} = \frac{\mathrm{d}^2 g(x+ct)}{\mathrm{d}(x+ct)^2} - \frac{c^2}{c^2}\frac{\mathrm{d}^2 g(x+ct)}{\mathrm{d}(x+ct)^2} = 0.$$

The wave equation is linear and homogeneous, so if $f(x-ct)$ is a solution and $g(x+ct)$ is a solution, then so is their sum!