Exercise 6.1 (Volumes) *Set up and evaluate integrals to calculate the volume of an* $L_x \times L_y \times L_z$ *rectangular solid, a cylinder of radius* R *and height* h*, and a sphere of radius* R*.*

$$
\begin{aligned}
V &= \int_0^{L_z} \int_0^{L_y} \int_0^{L_x} \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = L_x L_y L_z, \\
V &= \int_0^{h} \int_0^{2\pi} \int_0^{R} r \, \mathrm{d}r \, \mathrm{d}\phi \, \mathrm{d}z = \pi R^2 h, \\
V &= \int_0^{2\pi} \int_0^{\pi} \int_0^{R} r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi = \frac{4}{3}\pi R^3.
\end{aligned}
$$

6.5.2 Application

Exercise 6.2 (Earth) *Redo the cylinder and sphere volume calculations from exercise* **??**, *but with all integrals and integrands expressed* in earth coordinates. *Hint: It is easiest to consider the cylinder as sitting with its base on the origin (rather than centered on the origin). Then break up the integral into two parts. First, there is the conic section that extends from $\vartheta = \tan^{-1}(h/R)$ to $\vartheta = \pi/2$ and is bounded at the surface of the top of the cylinder. This surface can be described by $\mathfrak{z} = h/\sin\vartheta$. The other surface to consider is the outer shell of the cylinder. This surface can be described by the function $\mathfrak{z} = R/\cos\vartheta$, and it is relevant for $\vartheta = 0$ to $\vartheta = \tan^{-1}(h/R)$.*

Rectangular Solid: The earth coordinates are based on a sphere of radius r_0, which is supposed to be the mean radius of the earth. We can make this problem a little cleaner by assuming that the radius of our cylinder is much larger than the r_0, and therefore just using $r_0 = 0$. It is also easiest to consider the cylinder as sitting with its base on the origin (rather than centered on the origin). We break up the integral into two parts. First, there is the conic section that extends from $\vartheta = \tan^{-1}(h/R)$ to $\vartheta = \pi/2$ and is bounded at the surface of the top of the cylinder. This surface can be described by $\mathfrak{z} = h/\sin\vartheta$. The other surface to consider is the outer shell of the cylinder. This surface can be described by the function $\mathfrak{z} = R/\cos\vartheta$, and it is relevant for $\vartheta = 0$ to $\vartheta = \tan^{-1}(h/R)$. Thus,

$$
\begin{aligned}
V &= \int_0^{\tan^{-1}(h/R)} \int_0^{2\pi} \int_0^{R/\cos\vartheta} \mathfrak{z}^2 \cos\vartheta \, \mathrm{d}\mathfrak{z} \, \mathrm{d}\phi \, \mathrm{d}\vartheta + \int_{\tan^{-1}(h/R)}^{\pi/2} \int_0^{2\pi} \int_0^{h/\sin\vartheta} \mathfrak{z}^2 \cos\vartheta \, \mathrm{d}\mathfrak{z} \, \mathrm{d}\phi \, \mathrm{d}\vartheta, \\
&= 2\pi \int_0^{\tan^{-1}(h/R)} \int_0^{R/\cos\vartheta} \mathfrak{z}^2 \cos\vartheta \, \mathrm{d}\mathfrak{z} \, \mathrm{d}\vartheta + 2\pi \int_{\tan^{-1}(h/R)}^{\pi/2} \int_0^{h/\sin\vartheta} \mathfrak{z}^2 \cos\vartheta \, \mathrm{d}\mathfrak{z} \, \mathrm{d}\vartheta, \\
&= \frac{2\pi}{3} \int_0^{\tan^{-1}(h/R)} \frac{R^3}{\cos^2\vartheta} \, \mathrm{d}\vartheta + \frac{2\pi}{3} \int_{\tan^{-1}(h/R)}^{\pi/2} \frac{h^3 \cos\vartheta}{\sin^3\vartheta} \, \mathrm{d}\vartheta, \\
&= \frac{2\pi R^3}{3} \left(\tan\vartheta \Big|_0^{\tan^{-1}(h/R)} \right) + \frac{2\pi h^3}{3} \left(\frac{-1}{2\tan^2\vartheta} \Big|_{\tan^{-1}(h/R)}^{\pi/2} \right), \\
&= \frac{2\pi R^3}{3} \left(\frac{h}{R} \right) + \frac{2\pi h^3}{3} \left(0 + \frac{1}{2h^2/R^2} \right), \\
&= \frac{2\pi R^2 h}{3} + \frac{\pi R^2 h}{3}, \\
&= \pi R^2 h.
\end{aligned}
$$

Sphere: The sphere is much easier, because it is a surface of uniform $\mathfrak{z}$. Let's choose $r_0 = R$, then

$$
\begin{aligned}
V &= \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \int_{-R}^{0} (\mathfrak{z} + R)^2 \cos\vartheta \, \mathrm{d}\mathfrak{z} \, \mathrm{d}\phi \, \mathrm{d}\vartheta, \\
&= \frac{2\pi R^3}{3} \int_{-\pi/2}^{\pi/2} \cos\vartheta \, \mathrm{d}\vartheta, \\
&= \frac{4\pi R^3}{3}.
\end{aligned}
$$

Exercise 6.4 *Problem 5.4.25 Boas (2006). The volume inside a sphere of radius r is $V = \frac{4}{3}\pi r^3$. Then $\mathrm{dV} = 4\pi \mathrm{r}^2 \mathrm{dr} = \mathrm{Adr}$, where A is the area of the sphere. What is the geometrical meaning of the fact that the derivative of the volume is the area? Could you use this fact to find the volume formula given the area formula?*

The volume of a sphere is $V = \frac{4}{3}\pi r^3$, so $\mathrm{d}V = 4\pi r^2\,\mathrm{d}r = A\,\mathrm{d}r$. The volume of a sphere is related to the integral, with respect to r, of the surfaces of the spherical shells inside it. You can find the volume from the area by integrating $V = \int \mathrm{d}V = \int_0^r A\,\mathrm{d}r$.

Exercise 7.2 (Sines, Cosines, Exponentials) *Problem 7.5.12 of Boas (2006). Show that in (5.2) the average values of $\sin(mx)\sin(nx)$ and of $\cos(mx)\cos(nx)$, $m \neq n$, are zero (over a period), by using the complex exponential forms for the sines and cosines as in (5.3).*

$$\begin{aligned}
\langle \sin nx \sin mx \rangle &= -\frac{1}{4}\left\langle \left(e^{inx} - e^{-inx}\right)\left(e^{imx} - e^{-imx}\right)\right\rangle, \\
&= -\frac{1}{4}\left\langle e^{inx+imx} - e^{-inx+imx} - e^{inx-imx} + e^{-inx-imx}\right\rangle, \\
&= \begin{cases} -\frac{1}{4}(1-0-0+1) & \text{if } m = -n \neq 0 \\ -\frac{1}{4}(0-1-1+0) & \text{if } m = n \neq 0 \\ -\frac{1}{4}(0-0-0+0) & \text{if } m \neq \pm n \\ -\frac{1}{4}(1-1-1+1) & \text{if } m = n = 0 \end{cases} \quad = \begin{cases} 0, & \text{if } m = n = 0, \\ 0, & \text{if } m \neq n, \\ \frac{1}{2}, & \text{if } m = n \neq 0. \end{cases}
\end{aligned}$$

The last equality takes advantage of the oddness of sine.

$$\begin{aligned}
\langle \sin nx \sin mx \rangle &= \frac{1}{4}\left\langle \left(e^{inx} + e^{-inx}\right)\left(e^{imx} + e^{-imx}\right)\right\rangle, \\
&= \frac{1}{4}\left\langle e^{inx+imx} + e^{-inx+imx} + e^{inx-imx} + e^{-inx-imx}\right\rangle, \\
&= \begin{cases} \frac{1}{4}(1+0+0+1) & \text{if } m = -n \neq 0 \\ \frac{1}{4}(0+1+1+0) & \text{if } m = n \neq 0 \\ \frac{1}{4}(0+0+0+0) & \text{if } m \neq \pm n \\ \frac{1}{4}(1+1+1+1) & \text{if } m = n = 0 \end{cases} \quad = \begin{cases} 1, & \text{if } m = n = 0, \\ 0, & \text{if } m \neq n, \\ \frac{1}{2}, & \text{if } m = n \neq 0. \end{cases}
\end{aligned}$$

Quod erat demonstrum.

Exercise 7.3 (Derivatives) *a) Show that the following function $f(x)$ and Fourier series $g(x)$ are equivalent on the interval from $-\pi$ to π up to order of $\sin(2x)$. To do so, multiply the $f(x)$ and $g(x)$ functions by each of the following in turn: $\sin(x), \sin(2x)$ and $\cos(0x), \cos(x), \cos(2x)$. Show that the average value of the product from $-\pi$ to π is the same, for example that $\langle f(x)\sin(2x)\rangle = \langle g(x)\sin(2x)\rangle$. (see Boas, 2006, pg. 351).*

$$\begin{aligned}
&\forall : -\pi \leq x \leq \pi, \\
&f(x) = x(\pi - x)(\pi + x), \\
&g(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} 12\sin(nx).
\end{aligned}$$

b) Take the first derivative of $f(x)$ and $g(x)$ (by taking the derivative of the generic term in the series). Show that the resulting derivatives are equivalent, using the same method as in a).

(a) Exploiting the fact that any odd function will be zero on average, and explicitly calculating the

even functions, we find

$$\begin{aligned}
\langle f(x)\rangle &= \langle x(\pi - x)(\pi + x)\rangle = 0, & \langle g(x)\rangle &= 0. \\
\langle f(x)\cos x\rangle &= \langle x(\pi - x)(\pi + x)\cos x\rangle = 0, & \langle g(x)\cos x\rangle &= 0. \\
\langle f(x)\cos mx\rangle &= \langle x(\pi - x)(\pi + x)\cos mx\rangle = 0, & \langle g(x)\cos mx\rangle &= 0. \\
\langle f(x)\sin x\rangle &= \langle x(\pi - x)(\pi + x)\sin x\rangle & \langle g(x)\sin x\rangle &= 6 \\
&= \langle(\pi^2 x - x^3)\sin x\rangle = \pi^2 - (-6 + \pi^2) = 6 \\
\langle f(x)\sin mx\rangle &= \langle x(\pi - x)(\pi + x)\sin mx\rangle & \langle g(x)\sin mx\rangle &= \frac{(-1)^{m-1}}{m^3}6. \\
&= \langle(\pi^2 x - x^3)\sin nx\rangle = \frac{-6\cos m\pi}{n^3} = \frac{-6(-1)^m}{m^3}
\end{aligned}$$

(b) Now we check the derivative...

$$\begin{aligned}
&f'(x) = (\pi^2 - 3x^2), \\
&g'(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} 12\cos(nx).
\end{aligned}$$

$$\begin{aligned}
\langle f'(x)\rangle &= \langle(\pi^2 - 3x^2)\rangle = \pi^2 - \pi^2 = 0, & \langle g'(x)\rangle &= 0. \\
\langle f'(x)\cos x\rangle &= \langle(\pi^2 - 3x^2)\cos x\rangle = 0 - (-6), & \langle g'(x)\cos x\rangle &= 6. \\
\langle f'(x)\cos mx\rangle &= \langle(\pi^2 - 3x^2)\cos mx\rangle & \langle g'(x)\cos mx\rangle &= \frac{-6(-1)^m}{m^2}. \\
&= 0 - \frac{6\cos m\pi}{m^2} = \frac{-6(-1)^m}{m^2} \\
\langle f'(x)\sin x\rangle &= 0 & \langle g'(x)\sin x\rangle &= 0 \\
\langle f'(x)\sin mx\rangle &= 0 & \langle g'(x)\sin mx\rangle &= 0.
\end{aligned}$$

Quod erat demonstrum.