6.3.17 of Boas (exercise 8.1):

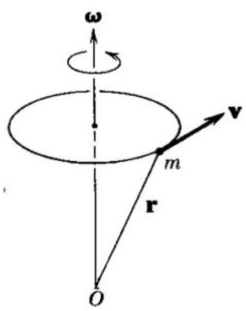

Figure 3.8

17. Expand the triple product for $\mathbf{a} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ given in the discussion of Figure 3.8. If $\mathbf{r}$ is perpendicular to $\boldsymbol{\omega}$ (Problem 16), show that $\mathbf{a} = -\omega^2\mathbf{r}$, and so find the elementary result that the acceleration is toward the center of the circle and of magnitude v^2/r.

6.3.17: Expand the triple product $\mathbf{a} = \omega \times \omega \times \mathbf{r}$ given in the discussion of Fig. 3.8. From (??), and if $\mathbf{r}$ is perpendicular to ω, we have

$$\mathbf{a} = \omega \times (\omega \times \mathbf{r}) = \omega\,(\omega \cdot \mathbf{r}) - \mathbf{r}\,(\omega \cdot \omega)\,,$$
$$= -\mathbf{r}\,(\omega \cdot \omega) = -\mathbf{r}\omega^2 = -\hat{\mathbf{r}}\frac{\omega^2 r^2}{r} = -\hat{\mathbf{r}}\frac{v_\phi^2}{r}.$$

8.3:

Compute the divergence and curl of $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$.

$$\nabla \cdot \mathbf{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3,$$

$$\nabla \times \mathbf{r} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \hat{\mathbf{i}}(0) + \hat{\mathbf{j}}(0) + \hat{\mathbf{k}}(0) = 0.$$

8.6

There are two ways to solve this problem. The first uses the gradient theorem. We define "uphill" as walking in a direction opposing a force (i.e., work is being done by you). Therefore, the amount of work (average walking uphill) from point $\mathbf{a}$ to point $\mathbf{b}$ is

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F}\cdot d\mathbf{l} = \int_{\mathbf{a}}^{\mathbf{b}} -\nabla\phi \cdot d\mathbf{l} = \phi(\mathbf{a}) - \phi(\mathbf{b}).$$

And if you are going in a roundtrip to school and back, then $\mathbf{a} = \mathbf{b}$, so no work is done and you cannot have gone uphill both ways. To put it another way,

$$0 = \phi(\mathbf{a}) - \phi(\mathbf{b}) - \phi(\mathbf{b}) + \phi(\mathbf{a}) = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F}\cdot d\mathbf{l} + \int_{\mathbf{b}}^{\mathbf{a}} \mathbf{F}\cdot d\mathbf{l}.$$

So, if either of the last two is negative (i.e., uphill on average), then the other must be positive (downhill on average).

The second way to do this is to consider the curl theorem over any closed path from school to home and back.

$$\oint \mathbf{F}\cdot d\mathbf{l} = \iint \nabla\times\mathbf{F}\cdot\hat{\mathbf{n}}\, dA = -\iint \nabla\times\nabla\phi\cdot\hat{\mathbf{n}}\, dA = 0.$$

The last is true because the curl of any gradient is zero. The first integral can be interpreted as the average amount of uphill over the whole circuit from home to school.
Notice that *neither of these solutions depends on the path chosen*, which is nice for debunking the storyteller (well, on the path we took to school...).
If there is a nonconservative force, then we can modify the Stokes theorem version trivially,

$$\oint \mathbf{F}\cdot d\mathbf{l} = \iint \nabla\times\mathbf{F}\cdot\hat{\mathbf{n}}\, dA = -\iint \nabla\times(\nabla\phi + \mathbf{F}_{nc})\cdot\hat{\mathbf{n}}\, dA = -\iint \nabla\times\mathbf{F}_{nc}\cdot\hat{\mathbf{n}}\, dA.$$

The nonconservative force might (or might not, depending on the path) contribute a nonzero amount of work required. Typical nonconservative forces are like walking through molasses, in which case the force always opposes the motion.

$$(a)\nabla\times\mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = 2\hat{\mathbf{k}}.,$$

$$(b) Cylindrical: \int_0^{2\pi}\int_0^1 2\hat{\mathbf{k}}\cdot\hat{\mathbf{n}} r\, dr\, d\phi = \int_0^{2\pi}\int_0^1 2r\, dr\, d\phi = 2\pi,$$

$$(c) Spherical: \int_0^{2\pi}\int_0^{\pi/2} 2\hat{\mathbf{k}}\cdot\hat{\mathbf{n}}\, \cancel{r^2}^{1} \sin\theta\, d\theta\, d\phi = \int_0^{2\pi}\int_0^{\pi/2} 2\cos\theta\sin\theta\, d\theta\, d\phi = \int_0^{2\pi} d\phi = 2\pi,$$

$$(d) Either: \oint \mathbf{v}\cdot d\mathbf{l} = \int_0^{2\pi} \mathbf{v}\cdot\hat{\phi} r\, d\phi = \int_0^{2\pi} (-r\sin\phi[\hat{\mathbf{r}}\cos\phi - \hat{\phi}\sin\phi] + r\cos\phi[\hat{\mathbf{r}}\sin\phi + \hat{\phi}\cos\phi])\cdot\hat{\phi} r\, d\phi,$$

as $\hat{\mathbf{i}} = \hat{\mathbf{r}}\cos\phi - \hat{\phi}\sin\phi, \quad \hat{\mathbf{j}} = \hat{\mathbf{r}}\sin\phi + \hat{\phi}\cos\phi \quad x = r\cos\phi, \quad y = r\sin\phi,$

$$\oint \mathbf{v}\cdot d\mathbf{l} = \int_0^{2\pi} (r\sin^2\phi + r\cos^2\phi) r\, d\phi = 2\pi.$$

8.7

8.8

$$(a) \nabla \cdot \mathbf{v} = 0.$$
$$(b) \oiint \mathbf{v} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \iiint \cancel{\nabla \cdot \mathbf{v}}^{0}\, \mathrm{d}V = 0.$$

(c) Not much. If the surface integral is open, there can be a large flow out through it, which can allow a large inflow elsewhere through the part of the surface you are integrating over. Think of puncturing a water balloon with a needle–a lot of flow can leave through a tiny hole even though the water is nearly incompressible and most of the surface integral (everywhere not punctured) is zero.