Chapter 9 HW key:

### 9.7.1 Manipulation

**Exercise 9.1** *Problem 8.2.1 of Boas (2006). For the following differential equation, separate variables and find a solution containing one arbitrary constant. Then find the value of the constant to give a particular solution satisfying the given boundary condition.*

$$xy' = y \tag{9.52}$$

*You do not need to compare to a computer solution.*

$$\begin{aligned} x\frac{\mathrm{d}y}{\mathrm{d}x} &= y, \\ \frac{\mathrm{d}y}{y} &= \frac{\mathrm{d}x}{x}, \\ \int \frac{\mathrm{d}y}{y} &= \int \frac{\mathrm{d}x}{x}, \\ \ln y = \ln x + \ln C&, \\ \ln y = \ln Cx&, \\ y = Cx&. \end{aligned}$$

Where $C$ is an arbitrary constant. The second step is to evaluate $C$ such that $y = 3$ when $x = 2$, thus

$$y = \frac{3}{2}x$$

**Exercise 9.2** *Problem 8.5.1 of Boas (2006). Solve the following differential equation by the methods discussed in Boas. You do not need to compare to a computer solution.*

$$y'' + y' - 2y = 0 \tag{9.53}$$

$$\begin{aligned} \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 0&, \\ y = Ae^{mx}&, \\ m^2 + m - 2 = 0 \to m = \frac{-1 \pm \sqrt{1+8}}{2} = -2 \text{ or} 1&, \\ y = Ae^{-2x} + Be^{1x}&. \end{aligned}$$

Now, to add the boundary conditions (y(0)=1, y'(0)=0), we note

y(0)=1=A+B
y'(0)=0=-2A+B

Subtracting the second equation from the first, we find:
1=3A

Thus, A=1/3
And then by substitution into the second equation
B=2/3

**Exercise 9.3** *Show that the general solution to the first-order, linear differential equation on pg. 401 of (Boas, 2006) is the same as the guess & check solution in (10.1) of these notes.*

The solution in (10.1) is

$$\langle \mathcal{P} \rangle \approx \langle \mathcal{P}_{ss} \rangle + Ae^{-t\frac{V_{out}}{\text{Vol.}}} = \langle \mathcal{P}_{ss} \rangle \left(1 - e^{-t\frac{V_{out}}{\text{Vol.}}}\right)$$

So, let's work on the one in pg. 401 of (Boas, 2006). The problem is stated as

$$y' + Py = Q.$$

The solution is formally,

$$\begin{aligned} ye^I &= \int Qe^I \, \mathrm{d}x + c, \qquad \text{or} \\ y &= e^{-I} \int Qe^I \, \mathrm{d}x + ce^{-I}, \qquad \text{where} \\ I &= \int P \, \mathrm{d}x. \end{aligned}$$

Our problem is

$$\frac{d}{dt}\langle \mathcal{P} - \mathcal{P}_{ss} \rangle + \frac{V_{out}}{\text{Vol.}}\langle \mathcal{P} - \mathcal{P}_{ss} \rangle \approx 0$$

Thus,

$$\begin{aligned} y &= \langle \mathcal{P} - \mathcal{P}_{ss} \rangle, \\ x &= t, \\ P &= \frac{V_{out}}{\text{Vol.}}, \\ Q &= 0, \\ I &= \int \frac{V_{out}}{\text{Vol.}} \, \mathrm{d}t = \frac{V_{out}t}{\text{Vol.}} \end{aligned}$$

Thus, the solution is

$$y = \langle \mathcal{P} - \mathcal{P}_{ss} \rangle = e^{-I} \int Qe^I \, \mathrm{d}x + ce^{-I} = ce^{-\frac{V_{out}}{\text{Vol.}}t} = Ae^{-t\frac{V_{out}}{\text{Vol.}}}$$

Which is the same as the solution given in (10.1).

**Exercise 9.4** *Find the leading order differential equation, by Taylor series expansion of the radiation forcing, to the Energy Balance Model of Section 9.2.3 for the remaining steady state solution (near $T_{ss3} = 174.438\,\mathrm{K}$). Solve this differential equation, and decide if this third steady state is a stable or unstable steady state solution.*

Evaluating the Taylor series near $T_{ss3}$,

$$\begin{aligned}
R_i &= 51.375\,\mathrm{W\,m^{-2}}\,(1+\dots)\,,\\
-R_o &= 51.375\,\mathrm{W\,m^{-2}}\left(-1-0.02218(T-T_{ss3})\right),\\
\frac{\mathrm{d}T}{\mathrm{d}t} &\approx \underbrace{\frac{-1.1395\,\mathrm{W\,m^{-2}\,K^{-1}}}{c}}_{\text{neg. feedback}}(T-T_{ss3})+\dots\\
T-T_{ss3} &\approx \Delta T e^{-t/\tau}, \qquad \tau = \frac{1.1395\,\mathrm{W\,m^{-2}\,K^{-1}}}{c}
\end{aligned}$$

This steady-state solution is stable.

### 9.7.3 Evaluate & Create

**Exercise 9.5** *Verify that the sum of the characteristic solution (9.49) and the particular solution (9.44) constitute a solution to the original equations for the two sliding masses (9.43).*

$$\begin{aligned}
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} x_{p1} \\ x_{p2} \end{bmatrix} + \begin{bmatrix} x_{c1} \\ x_{c2} \end{bmatrix},\\
\begin{bmatrix} x_{p1} \\ x_{p2} \end{bmatrix} &= \begin{bmatrix} Vt-D-2\mu S/k \\ Vt-2D-3\mu S/k \end{bmatrix},\\
\begin{bmatrix} x_{c1} \\ x_{c2} \end{bmatrix} &= (Ae^{i\omega_1 t}+Be^{-i\omega_1 t})\begin{bmatrix} 1+\sqrt{5} \\ -2 \end{bmatrix} + \left(Ce^{i\omega_2 t}+De^{-i\omega_2 t}\right)\begin{bmatrix} 1-\sqrt{5} \\ -2 \end{bmatrix},\\
\omega_1 &= \sqrt{\frac{k}{m}}\sqrt{\frac{3+\sqrt{5}}{2}}, \qquad \omega_2 = \sqrt{\frac{k}{m}}\sqrt{\frac{3-\sqrt{5}}{2}}
\end{aligned}$$

So,

$$\begin{aligned}
x_1 &= Vt-D-2\mu S/k+\left[1+\sqrt{5}\right](Ae^{i\omega_1 t}+Be^{-i\omega_1 t})+\left[1-\sqrt{5}\right]\left(Ce^{i\omega_2 t}+De^{-i\omega_2 t}\right),\\
x_2 &= Vt-2D-3\mu S/k-2(Ae^{i\omega_1 t}+Be^{-i\omega_1 t})-2\left(Ce^{i\omega_2 t}+De^{-i\omega_2 t}\right).
\end{aligned}$$

Our overall governing equations are:

$$\begin{aligned}
m\,\frac{\mathrm{d}^2x_1}{\mathrm{d}t^2} &= \left(k\,[Vt-x_1-D]-k\,[x_1-x_2-D]-\mu S\right),\\
m\,\frac{\mathrm{d}^2x_2}{\mathrm{d}t^2} &= \left(k\,[x_1-x_2-D]-\mu S\right).
\end{aligned}$$

From our solutions, we have

$$
\begin{aligned}
\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} &= -\omega_1^2 \left[1+\sqrt{5}\right](Ae^{i\omega_1 t} + Be^{-i\omega_1 t}) - \omega_2^2 \left[1-\sqrt{5}\right]\left(Ce^{i\omega_2 t} + De^{-i\omega_2 t}\right), \\
\frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} &= 2\omega_1^2 (Ae^{i\omega_1 t} + Be^{-i\omega_1 t}) + 2\omega_2^2 \left(Ce^{i\omega_2 t} + De^{-i\omega_2 t}\right), \\
k[Vt - x_1 - D] &= 2\mu S - k\left[1+\sqrt{5}\right](Ae^{i\omega_1 t} + Be^{-i\omega_1 t}) - k\left[1-\sqrt{5}\right]\left(Ce^{i\omega_2 t} + De^{-i\omega_2 t}\right), \\
k\left[x_1 - x_2 - D\right] &= \mu S + k\left[3+\sqrt{5}\right](Ae^{i\omega_1 t} + Be^{-i\omega_1 t}) + k\left[3-\sqrt{5}\right]\left(Ce^{i\omega_2 t} + De^{-i\omega_2 t}\right).
\end{aligned}
$$

$$
\begin{aligned}
\left(k\left[Vt - x_1 - D\right] - k\left[x_1 - x_2 - D\right] - \mu S\right) &= -k\left[4+2\sqrt{5}\right](Ae^{i\omega_1 t} + Be^{-i\omega_1 t}) - k\left[4-2\sqrt{5}\right]\left(Ce^{i\omega_2 t} + De^{-i\omega_2 t}\right), \\
\left(k\left[x_1 - x_2 - D\right] - \mu S\right) &= k\left[3+\sqrt{5}\right](Ae^{i\omega_1 t} + Be^{-i\omega_1 t}) + k\left[3-\sqrt{5}\right]\left(Ce^{i\omega_2 t} + De^{-i\omega_2 t}\right), \\
\frac{3+\sqrt{5}}{2}(1+\sqrt{5}) &= 4+2\sqrt{5}, \\
\frac{3-\sqrt{5}}{2}(1-\sqrt{5}) &= 4-2\sqrt{5}.
\end{aligned}
$$

These results, when assembled, prove the governing equations are satisfied by the solutions found.