Chp 10 HW key

10.1

10.9.1 Manipulation

Exercise 10.1 *Locate the fixed points of the following one-dimensional functions and use linear stability analysis and phase planes ($\dot{x}$ vs. x) to categorize them. Consider only real values of a and x. a) $\dot{x} = a - e^{-x^2}$, b) $\dot{x} = ax - x^5$ (note that a can be of positive, zero, or negative–consider all 3).*

a) See Fig. 10.30. b) See Fig. 10.31

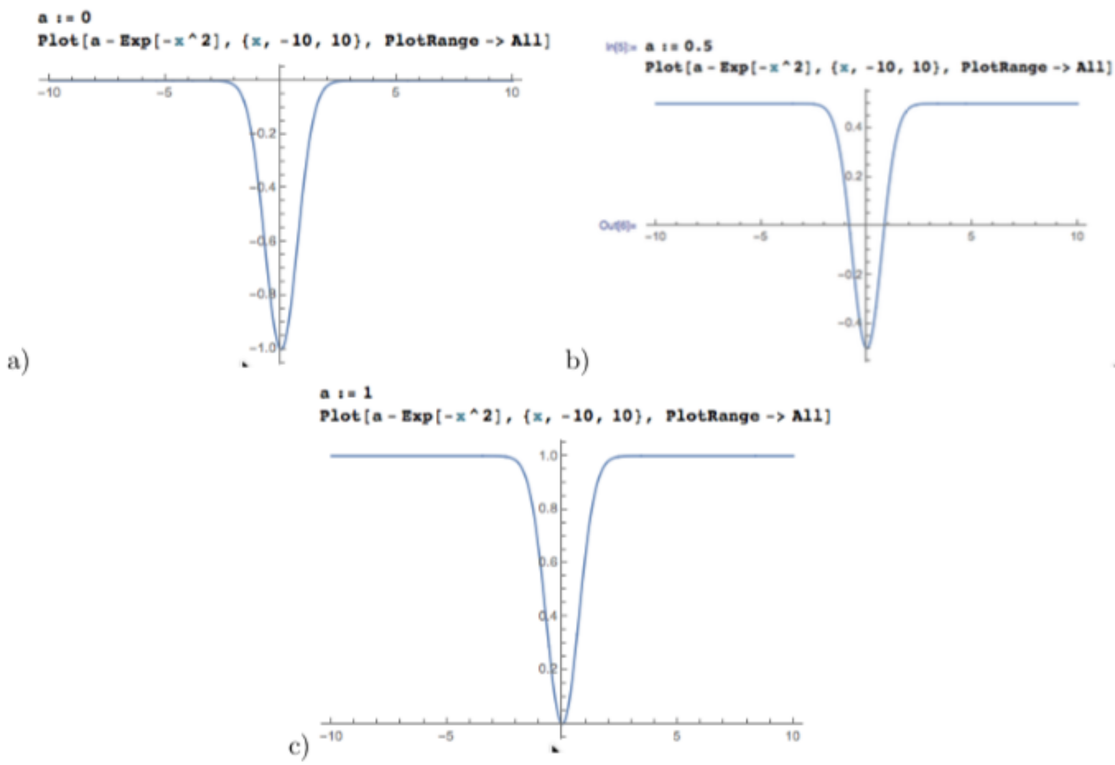

Figure 10.30: Plots of $\dot{x} = a - e^{-x^2}$ versus x varying a. In a), $a = 0$, where there are two fixed points at $\pm\infty$, with the one at $-\infty$ being stable and the one at ∞ being unstable. b) Shows when $a = 0.5$, where the fixed points are at $x = \pm\sqrt{-\ln a} = \pm 0.83255\ldots$. The one at $x < 0$ is stable and the other is unstable. c) Shows the one half-stable fixed point when $a = 1$ at $x = 0$. There are no real fixed points when $a > 1$ or $a < 0$.

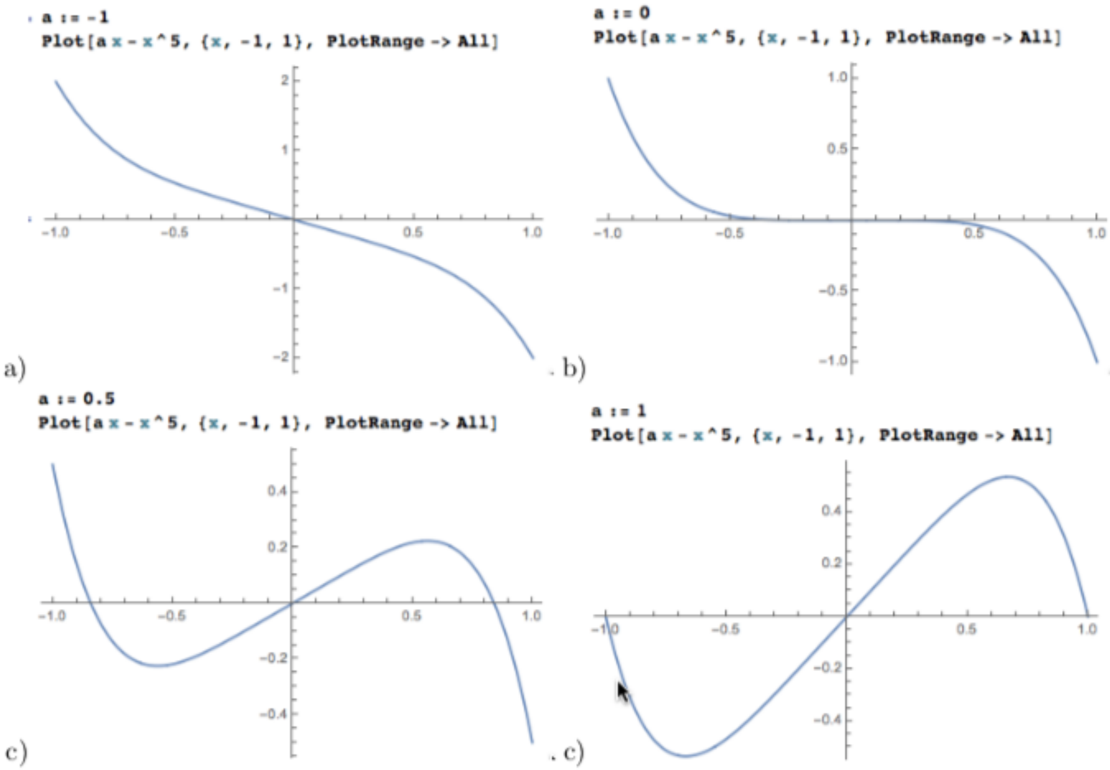

Figure 10.31: Plots of $\dot{x} = ax - x^5 = x(a - x^4)$ versus x varying a. In a), $a = -1$, where one stable root is at $x = 0$, with the rest being imaginary. b) Shows when $a = 0$, where all five roots are at $x = 0$, again this point is stable (although it is neutral–i.e., no slope–for the first 4 derivatives). c) Now there are three fixed points at $a = 0.5$. The one at $x = 0$ is unstable, and the other two are stable (pitchfork bifurcation). The two nonzero real roots are at $x = \pm\sqrt[4]{a}$. d) The solutions at $a = 1$ is the qualitatively the same as $a = 0.5$.

10.2

Exercise 10.2 *Draw the bifurcation diagrams (x^* vs. a) for the following functions as a varies: a) $\dot{x} = a - e^{-x^2}$, b) $\dot{x} = ax - x^5$.*

a) See Fig. 10.32. b) See Fig. 10.33

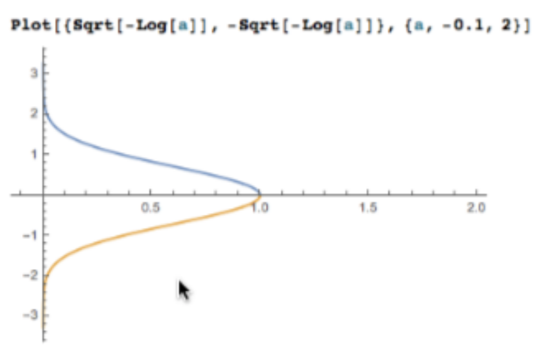

Figure 10.32: Bifurcation diagram of $0 = a - e^{-x^{*2}}$ versus a. Upper branch is unstable, lower is stable.

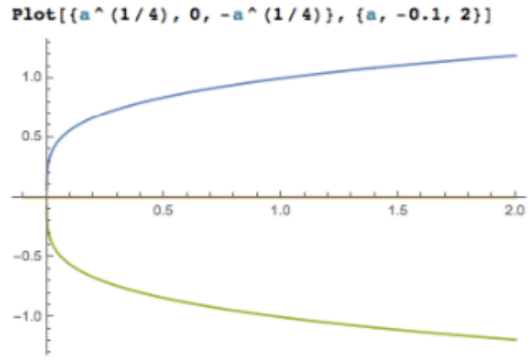

Figure 10.33: Bifurcation diagram of $0 = ax^* - x^{*5} = x^*(a - x^{*4})$ versus a. Nonzero branches are stable, branch at zero is stable for $a < 0$ and unstable for $a > 0$.

10.3

Exercise 10.3 *The Stommel (1961) model exhibits hysteresis (Section 10.4.6) when the forcing is varied. The Zaliapin and Ghil (2010) model (Section 9.2.3) also exhibits hysteresis when the incoming solar radiation is varied. Which of the the following systems has hysteresis as the parameter a is varied between -1 and 1? Why or why not?*

$$a) \quad \dot{x} = ax - x^3, \tag{10.94}$$
$$b) \quad \dot{x} = a + 2x - x^3, \tag{10.95}$$
$$c) \quad \dot{x} = a - x^2 \tag{10.96}$$

Actually, none of these exhibit hysteresis for a ranging between -1 and 1. What are the requirements for hysteresis? Multiple solutions in x for a given a (so that one can be close to one or the other as a is varying.), and a disappearance or reappearance of solutions with varying a that tends to make the solution jump from one branch to another in a different direction depending on whether a is increasing or decreasing. Let's take each in turn.

a) This is our prototypical pitchfork bifurcation. It has one stable root for $a < 0$ and two stable and one unstable for $a > 0$. So, the possibility exists that we might come in from $a > 0$ on the upper branch, transition onto the $x = 0$ branch for $a < 0$, and then leave on the lower branch. However, this supposes that we somehow can cross over the $x = 0$ stable or unstable fixed point. That is, if we arrive at $a = 0$ just above $x = 0$, there is no way to get to negative x and return on the bottom branch. So, no hysteresis.

b) This system is the closest to hysteresis among those presented. The parametric plot has the classic S shape that leads to hysteresis. However, the bluesky bifurcations that make the upper two branches or the lower two branches disappear occurs slightly outside of this value ($a = \sqrt{32/27} \approx 1.08866$). To locate this point, note that the extrema of the function in x occur at $x = \pm\sqrt{2/3}$ (set derivative $\frac{\partial \dot{x}}{\partial x} = 0$ and solve). The value where $a + 2x - x^3 = 0$ when $x = \pm\sqrt{2/3}$ occurs when $a = \pm\sqrt{32/27}$. However, if a is varied between, e.g., -1.1 and 1.1, this system will exhibit hysteresis.

c) This system has only two fixed points, and one is always unstable. Thus, there can be no hysteresis, as there are not two branches to jump between.

10.4

No real answers here, but looking forward to seeing what you find!