Chp 11 HW key

11.1

Exercise 11.1 *Problem 13.1.2 of Boas (2006). (a) Show that the expression* $u = \sin(x \pm vt)$ *describing a sinusoidal wave (see Chapter 7, Figure 2.3), satisfies the wave equation (1.4). Show that, in general,* $u = f(x - vt)$ *and* $u = f(x + vt)$ *satisfy the wave equation, where* f *is any function with a second derivative. This is the d'Alembert solution of the wave equation. (See Chapter 4, Section 11, Example 1.) The function* $f(x-vt)$ *represents a wave moving in the positive* x *direction and* $f(x+vt)$ *represents a wave moving in the opposite direction.*

(b) Show that $u(r,t) = (1/r)f(r - vt)$ *and* $u(r,t) = (1/r)f(r + vt)$ *satisfy the wave equation in spherical coordinates. [Use the first term of (7.1) for* $\nabla^2 u$ *since here* u *is independent of* θ *and* ϕ*.] These functions represent spherical waves spreading out from the origin or converging on the origin.*

a) First check the D'Alembert solutions,

$$u = \sin(x - vt) \to \frac{\partial^2 u}{\partial x^2} = -\sin(x - vt), \qquad \frac{\partial^2 u}{\partial t^2} = -v^2 \sin(x - vt), \qquad \therefore \nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}.$$

$$u = f(x - vt) \to \frac{\partial^2 u}{\partial x^2} = f''(x - vt), \qquad \frac{\partial^2 u}{\partial t^2} = v^2 f''(x - vt), \qquad \therefore \nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}.$$

$$u = f(x + vt) \to \frac{\partial^2 u}{\partial x^2} = f''(x + vt), \qquad \frac{\partial^2 u}{\partial t^2} = v^2 f''(x + vt), \qquad \therefore \nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}.$$

b) Now, check on the spherical form.

$$u = \frac{1}{r} f(r - vt) \to \nabla^2 u = ?$$

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial u}{\partial r}\right] = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(\frac{1}{r}f'(r - vt) - \frac{1}{r^2}f(r - vt)\right)\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left[rf'(r - vt) - f(r - vt)\right] = \frac{1}{r}f''(r - vt),$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{v^2}{r}f(r - vt), \qquad \therefore \nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}.$$

$$u = \frac{1}{r} f(r + vt) \to \nabla^2 u = ?$$

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial u}{\partial r}\right] = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(\frac{1}{r}f'(r + vt) - \frac{1}{r^2}f(r + vt)\right)\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left[rf'(r + vt) - f(r + vt)\right] = \frac{1}{r}f''(r + vt),$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{v^2}{r}f(r + vt), \qquad \therefore \nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}.$$

11.2

Exercise 11.2 *Problem 13.2.10 of Boas (2006). You do not need to make a computer plot of your results.*

Find the steady-state temperature distribution in a metal plate 10 cm square if one side is held at 100° and the other three sides at 0°. Find the temperature at the center of the plate. (The answer, but not the solution method, is written out in Boas).

Start from a general guess and check solution, similar to that in Section 11.2. Evaluate it on the boundaries. Note that while it is arbitrary which direction (x or y) is taken to be the oscillatory one, we can anticipate that the Fourier transform in x will simplify the complicated behavior in in transitioning from the zero temperatures to the 100 degree temperature on the one hot side.

$$\sum_k \left[A_{k+}e^{ikL} + A_{k-}e^{-ikL}\right]\left[B_{k+}e^{ky} + B_{k-}e^{-ky}\right] = 0, \tag{11.64}$$

$$\sum_k \left[A_{k+} + A_{k-}\right]\left[B_{k+}e^{ky} + B_{k-}e^{-ky}\right] = 0, \tag{11.65}$$

$$\sum_k \left[A_{k+}e^{ikx} + A_{k-}e^{-ikx}\right]\left[B_{k+}e^{kL} + B_{k-}e^{-kL}\right] = 0, \tag{11.66}$$

$$\sum_k \left[A_{k+}e^{ikx} + A_{k-}e^{-ikx}\right]\left[B_{k+} + B_{k-}\right] = 100. \tag{11.67}$$

By symmetry of the boundaries, we can anticipate only the Fourier components that will be equal on both sides of the plate. Based on where we choose the origin (x=0, x=L), this will turn out to be a sine series with periodicity on the interval from 0, L.

Considering (11.64-11.65), these are satisfied if $A_{k+} = -A_{k-} = A_k/(2i)$, producing a sine series, and $\sin(kL)$ to be zero, which means that $k = \frac{\pi}{L}, \frac{3\pi}{L}, \cdots = n\pi/L$ where n is odd.

$$A_k \sin(kx)\left[B_{k+}e^{kL} + B_{k-}e^{-kL}\right] = 0 \leftrightarrow B_{k+}e^{kL} + B_{k-}e^{-kL} = 0$$

$$\sum_k A_k \sin(kx)\left[B_{k+} + B_{k-}\right] = 100. \leftrightarrow B_{k+} + B_{k-} = 100,$$

$$B_{k+}e^{ky} + B_{k-}e^{-ky} = 100\frac{e^{k(L-y)} - e^{k(y-L)}}{e^{kL} - e^{-kL}} = 100\frac{\sinh(k(L-y))}{\sinh kL}$$

Now, we determine the A_k coefficients by Fourier's trick,

$$A_k = \frac{2}{L}\int_0^L \sin\left[\frac{n\pi x}{L}\right] = \frac{4}{n\pi}.$$

Some algebra combines these two equations for the remaining coefficients as

$$T = \sum_n \frac{400\sin(n\pi x/L)}{n\pi}\frac{\sinh(n\pi(L-y)/L)}{\sinh n\pi}.$$

Evaluating at the center of the plate gives

$$T = \sum_{\text{odd } n} \frac{400\sin(n\pi/2)}{n\pi}\frac{\sinh(n\pi/2)}{\sinh n\pi} = 25.3716 - 0.381232 + 0.00988551 + \cdots = 25.$$

Alternatively, you could have just argued that the vanishing Laplacian implies that there should be no anomalies, so the temperature at the center should be the average of the temperature on the four plates. This turns out to be correct, but I'm not sure how far you can push this from a rigorous perspective.

11.3

Exercise 11.3 *Problem 13.3.2 of Boas (2006). You do not need to make a computer plot of your results.*

A bar 10 cm long with insulated sides is initially at $100°$. *Starting at* $t = 0$, *the ends are held at* $0°$. *Find the temperature distribution in the bar at time* t. *(The answer, but not the solution method, is written out in Boas).*

As the sides are insulated, we do not expect any dependence on y, i.e., the temperature will be uniform in stripes across the bar, but we do expect dependences on x, t. Thus,

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial T}{\partial t}, \tag{11.68}$$

$$T = X_k(x)T_k(t), \tag{11.69}$$

$$\frac{\frac{\partial^2 X_k}{\partial x^2}}{X_k} = \frac{1}{\alpha^2}\frac{\frac{\partial T_k}{\partial t}}{T} = -k^2, \tag{11.70}$$

$$X_k = A_k \sin(kx) + B_k \cos(kx), \tag{11.71}$$

$$T_k = e^{-\alpha^2 k^2 t}. \tag{11.72}$$

The boundary conditions are $T = 0$ after $t = 0$, so we can consider only a sine series, with $k = n\pi/L$.

$$T = \sum_k A_k \sin(kx) e^{-\alpha^2 k^2 t}. \tag{11.73}$$

At $t = 0$, we can evaluate all of the A_k,

$$A_k = \frac{2}{L}\int_0^L 100 \sin\left[\frac{n\pi x}{L}\right] = \frac{400}{n\pi}. \tag{11.74}$$

For odd n. Thus,

$$T = \sum_{\text{odd } n} \frac{400 \sin(n\pi x/L)}{n\pi} e^{-(n\pi\alpha/L)^2 t}.$$

11.4

Exercise 11.4 *Write the Helmholtz equation in earth coordinates. Do you think the separable solutions will be the same or different from those found in spherical coordinates? Why or why not?*

To interpret the Helmholtz equation in earth coordinates, we just need to look up the Laplacian from (8.41). I'll use u for the Helmholtz unknown, so as not to confuse with longituyde ϕ.

$$\nabla^2 u + k^2 u = 0,$$

$$\frac{1}{(\mathfrak{z}+r_0)^2}\frac{\partial}{\partial \mathfrak{z}}\left((\mathfrak{z}+r_0)^2\frac{\partial u}{\partial \mathfrak{z}}\right)+\frac{1}{(\mathfrak{z}+r_0)^2\cos^2\vartheta}\frac{\partial^2 u}{\partial \phi^2}+\frac{1}{(\mathfrak{z}+r_0)^2\cos\vartheta}\frac{\partial}{\partial \vartheta}\left(\cos\vartheta\frac{\partial u}{\partial \vartheta}\right)+k^2u=0.$$

If we assume separable form for u, then

$$u=\sum Z(\mathfrak{z})P(\phi)T(\vartheta)$$

Each term in that series is a product of a function of height ($\mathfrak{z}$) times a function of longitude (ϕ) times a function of latitude (ϑ). Contrast this against the separable solution in spherical coordinates

$$u=\sum R(r)\Phi(\phi)\Theta(\theta)$$

Since there is a simple mapping from spherical to earth coordinates, $r \to r_0+\mathfrak{z}, \theta \to \vartheta - \pi/2$, which preserves a one-to-one functional relationship between one earth coordinate variable and one spherical coordinate variable, all of the coordinate surfaces are shared between earth coordinates and spherical coordinates. Since separable solutions are solutions where each variable can vary independently, and since the different variables are not mixed up between earth and spherical coordinates, we expect that there will be equivalent functions in earth coordinates (earth harmonics?) to the separable solutions in spherical coordinates (spherical harmonics). The only difference between these functions will be that the arguments will be processed to make the transformation $r \to r_0+\mathfrak{z}, \theta \to \vartheta - \pi/2$.

11.6

11.8.4 Scheming Schematics and Articulate Analysis

Exercise 11.6 *Look at* http://tinyurl.com/mljujml *and* http://tinyurl.com/ol3al47. *a) Contrast these against the separation of variables in the Cartesian coordinate cases. b) Why aren't the solutions sines and cosines? c) How can it matter which coordinate system we choose–that is, what is so special about separable solutions?*

a) The separable solutions to the Helmholtz equation in spherical and cylindrical coordinates differ from each other and from Cartesian coordinates. In Cartesian coordinates, we have sines, cosines,

and exponentials (or equivalently sinh and cosh). In parabolic cylindrical coordinates, the z axis has sines and cosines, as in Cartesian, but the u and v separable solutions are very different because of the Weber differential equations that result during separation. In the spherical case, θ has sines and cosines as $\frac{\partial^2}{\partial\phi^2}$ occurs without variable coefficients in the spherical version of Helmholtz, but both r and ϕ have non-constant coefficients in that system, so different separable solutions result–the Legendre polynomials (and the associated Legendre functions when r is retained).
b) The other functions that differ from sines and cosines occur because in the new coordinate systems, the coefficients of the derivatives are not constants. We know that a constant coefficient, homogeneous differential equation will have exponentials, sines, and cosines, but in these other coordinate systems the coefficients are not constant. Physically, these differences result from the curvature of the coordinate surfaces in space.
c) The overall solution to the differential equation is a *sum* over all separable solutions, such that any initial or boundary conditions supplied are satisfied. However, each term in the separable series can depend on the coordinate system, as they are found by the property that they are constant over different surfaces (spherical shells, cylinders, etc.). They are all orthogonal function sets, and in fact they are generally complete, so that we can convert between different representations in different coordinate systems, but only when the whole sum is retained. Each term in the series is not guaranteed to be represented by a single term in the series in a different coordinate system (just like our Taylor series versus Fourier series problem on the midterm).