12.7 Homework Problems

Exercise 12.1 (Materially Speaking) *Show that the material derivative of density (12.7) can be found by taking the total derivative with respect to time of the density, when the position being considered is also a function of time, i.e.,*

$$\frac{d}{dt}(\rho(x(t), y(t), z(t), t)) \tag{12.48}$$

by application of the chain rule. Note how the association of the partial derivatives of position coordinates with respect to time to the displacement velocities requires knowing that the position is a function of the material at hand.

$$\frac{d}{dt}(\rho(x(t), y(t), z(t), t)) = \frac{\partial \rho}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \rho}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \rho}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial \rho(x,y,z,t)}{\partial t} \tag{12.49}$$

$$= \frac{\partial \rho(x,y,z,t)}{\partial t} + \left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}\right) \cdot \nabla \rho(x, y, z, t) \tag{12.50}$$

The rate of change of position of the density on a material parcel is associated with the velocity, so

$$\frac{d}{dt}(\rho(x(t), y(t), z(t), t)) = \frac{\partial \rho(x,y,z,t)}{\partial t} + \mathbf{v} \cdot \nabla \rho(x, y, z, t) \tag{12.51}$$

This is only true because motion of density is related to motion of the medium.

Exercise 12.2 (Stressing Out) *Contrast the stress tensors (12.33)–(12.35) among the three types of continua: liquids, solids, and gasses. Clarify what is assumed (compressible?), what stress is proportional to, and the role of pressure in each stress tensor.*

$$\text{solid}: \quad \sigma_{ij} = -p\delta_{ij} + S_{ijkl}\frac{1}{2}\left(\nabla_k \Delta x_l + \nabla_l \Delta x_k\right), \tag{12.52}$$

$$\text{liquid}: \quad \sigma_{ij} = -p\delta_{ij} + 2\mu\frac{1}{2}\left(\nabla_i v_j + \nabla_j v_i\right), \tag{12.53}$$

$$\text{gas}: \quad \sigma_{ij} = (-p + \lambda D_{kk})\delta_{ij} + 2\mu D_{ij}. \tag{12.54}$$

The solid stress tensor contains a pressure contribution on the diagonal, and then a proportionality between the remaining stress and the strain (the symmetrized displacement gradient). Little is assumed about the solid. The liquid stress tensor also contains a pressure contribution on the diagonal, and the remainder relates a proportionality between the stress and the strain *rate* (the symmetrized velocity gradient). The gas stress tensor also contains a pressure contribution on the diagonal, and the remainder relates a proportionality between the stress and the strain *rate* (the symmetrized velocity gradient), *plus* a pressure like viscosity acting on the convergence/divergence of the gas velocity.

Exercise 12.3 (Phasers on Stun) *Relate your answers Exercise 12.2 to the statements: "Solid is the state in which matter maintains a fixed volume and shape; liquid is the state in which matter adapts to the shape of its container but varies only slightly in volume; and gas is the state in which matter expands to occupy the volume and shape of its container.*

The solid stress tensor resists displacements of parts of the solid from their original location, or at least strains, which are displacements that cannot be represented as a translation or rotation. The liquid stress tensor resists the rate of displacement/strain, which means that liquids may adjust to the shape of their container slowly without feeling a stress. The gas has a similar viscosity to the liquid one plus a viscosity on the divergence of the motions, which is missing the liquid form. The reason it is missing for the liquid is that this kind of motion (convergence or volume-altering) is not considered in the liquid phase to be very important.

Exercise 12.4 (Vector Covariant Fluids) *Examine the Euler equations of perfect fluid motion* (12.28)–(12.29)*. As previous chapters have emphasized, a partial differential equation consisting only of vectors, scalars and divergence, gradient, and curl will be vector covariant (consistent under rotations and rescaling of the reference frame and units). Are these equations vector covariant? Give an example of a term that could be added to violate this vector covariance.*

The Euler equations as written contain a variety of different quantities (scalars: ρ, vectors: velocity) plus the gradient of a scalar and the divergence of a vector.

$$\rho\frac{D\mathbf{v}}{Dt} = -\nabla p + \rho\mathbf{g}, \tag{12.55}$$

$$\tfrac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0. \tag{12.56}$$

The material derivative of the velocity might be a bit confusing, however. Is this vector covariant? Well, we know the material derivative of a scalar is vector covariant and produces a scalar, as (12.7) shows,

$$\tfrac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = \frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{v} = 0. \tag{12.57}$$

But, what about $\frac{D\mathbf{v}}{Dt}$? Well, note that beginning with the following, which has only vector covariant operations,

$$\tfrac{\partial(\rho\mathbf{v})}{\partial t} + \nabla\cdot(\rho\mathbf{v}\mathbf{v}) \tag{12.58}$$

Is vector covariant. This also is a bit confusing because the two velocities inside the parentheses are not combined with a dot product, but with an external product, or in index notation

$$\tfrac{\partial(\rho v_i)}{\partial t} + \nabla_i\cdot\left(\rho v_i v_j\right) \tag{12.59}$$

This means that the quantity inside the parentheses is a second-rank tensor, but that is tensor covariant and so is its divergence, which is a vector. Distribute out the derivatives and note,

$$\tfrac{\partial(\rho\mathbf{v})}{\partial t} + \nabla\cdot(\rho\mathbf{v}\mathbf{v}) = \mathbf{v}\underbrace{\left(\tfrac{\partial(\rho)}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right)}_{0} + \rho\left(\tfrac{\partial(\mathbf{v})}{\partial t} + \mathbf{v}\cdot\nabla(\mathbf{v})\right) \qquad = \rho\frac{D\mathbf{v}}{Dt}. \tag{12.60}$$

Thus, the material derivative is equal to a vector covariant operation and is vector covariant itself. What could be added to this equation to make it not vector covariant? Basically, any partial derivative or vector component on its own without its partners in one of the div, grad, or curl operations, e.g., $\frac{\partial\rho}{\partial x}$, u_x, etc.

Exercise 12.5 (Salty) *Equation* (12.5) *shows how the conservation of mass equation (a.k.a. the continuity equation or density equation) is related to a rate of change of mass in a volume plus a transport of density through the volume. Consider instead the conservation of a tracer* (12.12)*, neglecting diffusion ($\mu = 0$). If you integrate this equation over volume as in* (12.6)*, what does it relate to conservation of?*

Integrating the concentration equation, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho c \,\mathrm{d}V + \int_V \nabla \cdot (\rho c \mathbf{v}) \,\mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t} M_c + \oint_S \rho c \mathbf{v} \cdot \hat{\mathbf{n}} \,\mathrm{d}S = 0. \tag{12.61}$$

As c is the mass fraction, that is the proportion of the mass associated with the tracer constituent, the product of density (total mass/volume) times mass fraction is just the mass of the tracer constituent mass per unit volume M_c.