# Spring 2026 EEPS1820
# Homework 1, due Jan 30, 11:59PM

COVERING: Vallis Ch 1, 2;

## 1 Vallis (2019) Problem 1.1

1.1 Show that the derivative of an integral is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1(t)}^{x_2(t)} \varphi(x,t)\,\mathrm{d}x = \int_{x_1}^{x_2} \frac{\partial \varphi}{\partial t}\,\mathrm{d}x + \frac{\mathrm{d}x_2}{\mathrm{d}t}\varphi(x_2,t) - \frac{\mathrm{d}x_1}{\mathrm{d}t}\varphi(x_1,t). \qquad \text{(P1.1)}$$

By generalizing to three dimensions show that the material derivative of an integral of a fluid property is given by

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V \varphi(\boldsymbol{x},t)\,\mathrm{d}V = \int_V \frac{\partial \varphi}{\partial t}\,\mathrm{d}V + \int_S \varphi\boldsymbol{v}\cdot \mathrm{d}\boldsymbol{S} = \int_V \left[\frac{\partial \varphi}{\partial t} + \nabla\cdot(\boldsymbol{v}\varphi)\right]\mathrm{d}V, \qquad \text{(P1.2)}$$

where the surface integral ($\int_S$) is over the surface bounding the volume $V$. Hence deduce that

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V \rho\varphi\,\mathrm{d}V = \int_V \rho\frac{\mathrm{D}\varphi}{\mathrm{D}t}\,\mathrm{d}V. \qquad \text{(P1.3)}$$

You may assume Leibniz Integral Rule to begin (see Wikipedia for the rule and a proof).

## 2 Vallis (2019) Problem 1.2

1.2 *(a)* If molecules move quasi-randomly, why is there no diffusion term in the mass continuity equation?

*(b)* Suppose that a fluid contains a binary mixture of dry air and water vapour. Show that the change in mass of a parcel of air due to the diffusion of water vapour is exactly balanced by the diffusion of dry air in the opposite direction.

Hint: the diffusion equation for either of the two constituents is

$$\frac{D\rho_i}{Dt} + \rho_i\nabla\cdot\mathbf{v} = \kappa\nabla^2\rho_i, \qquad (1)$$

and their combined density is $\rho = \rho_a + \rho_w$.

# 3 Vallis (2019) Problem 1.5

1.5 Show that viscosity will dissipate kinetic energy in a compressible fluid.

Hint: Use integration by parts over the whole fluid volume with no-slip (i.e., velocity is zero on the boundaries), following Section 1.7.3. The correct compressible viscous Navier-Stokes momentum equation is

$$\rho \frac{D\mathbf{v}}{Dt} + \nabla p - \rho g = \mu \nabla^2 \mathbf{v} + \frac{1}{3}\mu \nabla(\nabla \cdot \mathbf{v})$$

# 4 Vallis (2019) Problem 2.2

2.2 *(a)* Show that on Earth we might normally expect the centrifugal term to be much larger than the Coriolis term. Show that if the centrifugal term is incorporated into gravity, and if Earth is a perfect sphere, then gravity is no longer in the local vertical. Estimate the angle by which the apparent gravity differs from the vertical.

*(b)* If Earth were a perfect sphere, but with otherwise the same distribution of continents and ocean basins, would the distribution of sea level be more or less the same as it is today or would it be radically different?

On Earth $|\Omega| \approx \frac{2\pi}{24hr}$ (approximately because the sidereal day is about 4 minutes less than the solar day which is just 24 hours).

a) Note that the local "vertical" Vallis uses here is not the "plumb line" definition of vertical which is the direction a motionless pendulum points, but the direction perpendicular to the surface.

# 5 Vallis (2019) Problem 2.7

2.7 Show that the inviscid, adiabatic, hydrostatic primitive equations for a compressible fluid conserve a form of energy (kinetic plus potential plus internal), and that the kinetic energy has no contribution from the vertical velocity. Obtain an explicit form for the conserved energy, and provide a physical interpretation for this result. (You may assume Cartesian geometry and a uniform gravitational field in the vertical direction.)

Alternatively, use the hydrostatic Boussinesq equations and again show that the vertical velocity does not contribute. Obtain an explicit form for the conserved energy and interpret your result.

Hint: follow the procedure from Section 1.7, but beginning with equations 2.41. You can use the tangent plane form, rather than the spherical (i.e., 2.43a and 2.43b instead of 2.41a and 2.41b). Also, you can use either the compressible form or the Boussinesq form–the latter will be more directly applicable in this class.

# 6 Vallis (2019) Problem 2.8

2.8 *(a)* Consider a scalar field, like temperature, $T$. Explain in words why the material derivative in a rotating frame is equal to the material derivative in the inertial frame; that is, explain why $(\mathrm{D}T/\mathrm{D}t)_I = (\mathrm{D}T/\mathrm{D}t)_R$.

*(b)* The material derivative of a scalar is given by $\partial T/\partial t + (\boldsymbol{v} \cdot \nabla)T$. Show (mathematically, with equations) that the individual terms are different in the rotating and inertial frames (and obtain an expression for how much) but that their sum is the same.

b) Hint: write out the velocity in each frame explicitly using (2.7): $\mathbf{v}_I = \mathbf{v}_R + \boldsymbol{\Omega} \times \mathbf{r}$

# References

Vallis, G. K. (2019). *Essentials of Atmospheric and Oceanic Dynamics.* Cambridge University Press.