Spring 2026 EEPS1820
Homework 5, due Mar 13, 11:59PM

COVERING: Vallis Ch 4, 5, 6, 8;

1 Vallis (2019) Problem 4.4

4.4 Linearize the f-plane shallow water system about a state of rest. Suppose that there is an initial disturbance given in the general form

$$\eta = \iint \widetilde{\eta}_{k,l} e^{i(kx+ly)} \, dk \, dl, \tag{P4.2}$$

where η is the deviation surface height and the Fourier coefficients $\widetilde{\eta}_{k,l}$ are given, and that the initial velocity is zero.

(a) Obtain the geopotential field at the completion of geostrophic adjustment, and show that the deformation scale is a natural length scale in the problem.

(b) Show that the change in total energy during the adjustment is always less than or equal to zero. Neglect any initial divergence.
N.B. Because the problem is linear, the Fourier modes do not interact.

2 Vallis (2019) Problem 4.5

4.5 If energy conservation is one of the most basic physical laws, how can energy be lost in geostrophic adjustment?

3 Vallis (2019) Problem 4.7

4.7 In the shallow water equations show that, if the flow is approximately geostrophically balanced, the energy at large scales is predominantly potential energy and the energy at small scales is predominantly kinetic energy. Define precisely what 'large scale' and 'small scale' mean in this context.

4 Vallis (2019) Problem 4.8

4.8 In the shallow water geostrophic adjustment problem, show that at large scales the velocity essentially adjusts to the height field, and that at small scales the height field essentially adjusts to the velocity field. Your derivation may be detailed and mathematical, but explain the result at the end in words and in physical terms.

5 Vallis (2019) Problem 5.1a

5.1 Do either or both:

(a) Carry through the derivation of the quasi-geostrophic system starting with the anelastic equations and obtain (5.66).

In each case, state the differences between your results and the Boussinesq result.

6 Vallis (2019) Problem 5.2

5.2 (a) The shallow water *planetary geostrophic* equations may be derived by simply omitting ζ in the equation

$$\frac{\mathrm{D}}{\mathrm{D}t}\frac{\zeta + f}{h} = 0 \qquad \text{(P5.1)}$$

by invoking a small Rossby number, so that ζ/f is small. We then relate the velocity field to the height field by hydrostatic balance and obtain:

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{f}{h}\right) = 0, \qquad fu = -g\frac{\partial h}{\partial y}, \quad fv = g\frac{\partial h}{\partial x}. \qquad \text{(P5.2)}$$

The assumptions of hydrostatic balance and small Rossby number are the same as those used in deriving the quasi-geostrophic equations. Explain nevertheless how some of the assumptions used for quasi-geostrophy are in fact different from those used for planetary-geostrophy, and how the derivations and resulting systems differ from each other. Use any or all of the momentum and mass continuity equations, scaling, nondimensionalization and verbal explanations as needed.

(b) Explain if and how your arguments in part (a) also apply to the stratified equations (using, for example, the Boussinesq equations or pressure coordinates).

7 Vallis (2019) Problem 5.4

5.4 Consider a wind stress imposed by a mesoscale cyclonic storm (in the atmosphere) given by

$$\boldsymbol{\tau} = -A\mathrm{e}^{-(r/\lambda)^2}(y\,\hat{\mathbf{i}} - x\,\hat{\mathbf{j}}), \tag{P5.3}$$

where $r^2 = x^2 + y^2$, and A and λ are constants. Also assume constant Coriolis gradient $\beta = \partial f/\partial y$ and constant ocean depth H. In the ocean, find

(a) the Ekman transport, (b) the vertical velocity $w_E(x, y, z)$ below the Ekman layer, (c) the northward velocity $v(x, y, z)$ below the Ekman layer and (d) indicate how you would find the westward velocity $u(x, y, z)$ below the Ekman layer.

8 Vallis (2019) Problem 5.5

5.5 In an atmospheric Ekman layer on the f-plane let us write the momentum equation as

$$\boldsymbol{f} \times \boldsymbol{u} = -\nabla\phi + \frac{1}{\rho_a}\frac{\partial \boldsymbol{\tau}}{\partial z}, \tag{P5.4}$$

where $\boldsymbol{\tau} = A\rho_a \partial \boldsymbol{u}/\partial z$ and A is a constant eddy viscosity coefficient. An *independent* formula for the stress at the ground is $\boldsymbol{\tau} = C\rho_a \boldsymbol{u}$, where C is a constant. Let us take $\rho_a = 1$, and assume that in the free atmosphere the wind is geostrophic and zonal, with $\boldsymbol{u}_g = U\hat{\mathbf{i}}$.

(a) Find an expression for the wind vector at the ground. Discuss the limits $C = 0$ and $C = \infty$. Show that when $C = 0$ the frictionally-induced vertical velocity at the top of the Ekman layer is zero.

(b) Find the vertically integrated horizontal mass flux caused by the boundary layer.

(c) When the stress on the atmosphere is $\boldsymbol{\tau}$, the stress on the ocean beneath is also $\boldsymbol{\tau}$. Why? Show how this is consistent with Newton's third law.

(d) Determine the direction and strength of the surface current, and the mass flux in the oceanic Ekman layer, in terms of the geostrophic wind in the atmosphere, the oceanic Ekman depth and the ratio ρ_a/ρ_o, where ρ_o is the density of the seawater. Include a figure showing the directions of the various winds and currents. How does the boundary-layer mass flux in the ocean compare to that in the atmosphere? (Assume, as needed, that the stress in the ocean may be parameterized with an eddy viscosity.)

Partial solution for (a): A useful trick in Ekman layer problems is to write the velocity as a complex number, $\hat{u} = u + \mathrm{i}v$ and $\hat{u}_g = u_g + \mathrm{i}v_g$. The fundamental Ekman layer equation may then be written as

$$A\frac{\partial^2 \widehat{U}}{\partial z^2} = \mathrm{i} f \widehat{U}, \tag{P5.5}$$

where $\widehat{U} = \hat{u} - \hat{u}_g$. The solution to this is

$$\hat{u} - \hat{u}_g = [\hat{u}(0) - \hat{u}_g]\exp\left[-\frac{(1+\mathrm{i})z}{d}\right], \tag{P5.6}$$

where $d = \sqrt{2A/f}$ and the boundary condition of finiteness at infinity eliminates the exponentially growing solution. The boundary condition at $z = 0$ is $\partial\hat{u}/\partial z = (C/A)\hat{u}$; applying this gives $[\hat{u}(0) - \hat{u}_g]\exp(\mathrm{i}\pi/4) = -Cd\hat{u}(0)/(\sqrt{2}A)$, from which we obtain $\hat{u}(0)$, and the rest of the solution follows.

References

Vallis, G. K. (2019). *Essentials of Atmospheric and Oceanic Dynamics.* Cambridge University Press.