Fall 2015 GEOL0350–GeoMath Midterm Exam

1 Bike Ride Home

When I ride my bike home, I have two choices of how to climb up to the top of College Hill after passing the Granoff Center on the Pembroke Path. Route #1 involves 1 flat block continuing along the Pembroke Path and 1 steep upward block (left up Meeting Street), while Route #2 involves 1 medium steep ascent left up Olive Street and 1 medium steep ascent up Brown Street. Both paths start in the same place (Granoff Center) and end in the same place (Brown and Meeting).

1.1 Potential Energy

I expend energy to gain potential energy ($E = mgz$) climbing up the hill on either path. Do I gain more potential energy going up the steeper path (Meeting Street)?

No, the potential energy gained is the same in either case.

1.2 Use the Force

The gradient of the potential energy gives the force of gravity $\mathbf{F}_g = -\nabla E$. Use this fact, and the gradient theorem, to relate the integrated force of gravity in the direction of motion going up Route #1 to that of Route #2. (Hint: mind the vectors!)

In both cases $E(b) - E(a) = \int_b^a \nabla E = -\int_b^a \mathbf{F}_g \cdot \mathrm{d}\mathbf{l}$

1.3 Frictionless

Now, riding to work, I can ride down either path. If all of my potential energy is converted into kinetic energy ($\frac{1}{2}mv^2$), and I begin at the top of the hill with negligible kinetic energy, which path will give me more speed when I reach the Granoff Center?

Equal kinetic energy will result from either path.

1.4 Frictional Gradient?

In practice, when I ride down, I tend to apply my brakes more on the steeper descent. Can my application of brakes be modeled by a force like $F_f = -\nabla\psi(z)$ for some function of height $\psi(z)$? No, because that would mean that the application of brakes would be equal along either path due to the gradient theorem above.

1.5 Frictional Function

How about modeling the application of brakes with $F_f = \nabla \times \mathbf{w}(z)$?

Sure. It is then possible for $\int_b^a \mathbf{F}_f \cdot \mathrm{d}\mathbf{l}$ to depend on which path is traveled. However, most choices of $\mathbf{w}$ would be inaccurate!

2 Schist

Figure 1: **A folded rock.**

The schist depicted has been twisted and folded, so that its original parallel sedimentary layers have been stretched, rotated, and squeezed like taffy. This problem will explore some possibilities of stretching and squeezing operations as represented by matrices.

2.1 Equations for a Line

Consider a vector (x_i, y_i) that is representing the original axis normal to the sedimentary layers of the schist, with a magnitude representing the compaction of the layers (e.g., the gradient of sediment age). After metamorphosis, the equivalent vector has been changed to (x_f, y_f).

$$ax_i + by_i = x_f, \tag{1}$$

$$cx_i + dy_i = y_f, \tag{2}$$

If the transformation $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is known, and its inverse is known $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$, can x_i, y_i be determined from x_f, y_f? Yes, multiply both sides by the inverted matrix.

2.2 Invertible, But Trivial

If $ax_i + by_i = 0, cx_i + dy_i = 0$, and if $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and its inverse $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$ are known, find x_i, y_i.

$$(x_i, y_i) = 0.$$

2.3 Escape!

Suppose $ax_i + by_i = 0, cx_i + dy_i = 0$, but you do not have the inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Is it possible that there might be an answer other than the one in problem 2.2? Yes, if the determinant of the matrix is zero, then it can not be inverted. At least one of the eigenvalues will be zero, and the corresponding eigenvector will satisfy this equation regardless of the values of x_i, y_i.

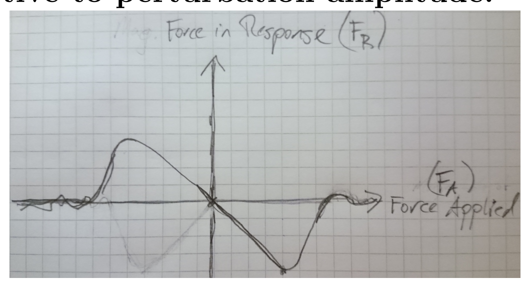
2.4 Eigenvectors, Eigenvalues

Suppose x_i, y_i is an eigenvector of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Describe the effect of the metamorphosis on the stratigraphy, that is, how does x_i, y_i compare to x_f, y_f? The two vectors are parallel, although they may be scaled by a constant factor. So, no twisting or rotating, just stretching and squeezing.

3 Rocking with Taylor Swiftly

Consider pushing on the rock depicted on the right. If you push or pull gently, the rock will push back. If you push or pull hard, not so much. We will use this example to consider how nonlinear functions are sensitive to amplitude in a way that linear functions are not.

Figure 2: **A situation that's sensitive to perturbation amplitude.**



3.1 Equal and Opposite

On the right, make a graph depicting the force applied, F_A (positive=push in horizontal direction, negative=pull in horizontal direction) versus the force back from the rock F_R (after equilibration). You can suppose that all forces go in the same direction. (Hint: prevent acceleration up to a point, and $F_A + F_R = ma$).

3.2 Constant Approximation

If the function $F_R(F_A)$ is fit with a Taylor series around $F_A = 0$ and truncated at the first (constant) term ($F_R(F_A) \approx c_0$). Describe this system's response to applied forces.

As $F_R(0) = c_0 = 0$, this system has no response, so $ma = F_A$.

3.3 Linear Approximation

If the Taylor series for $F_R(F_A)$ around $F_A = 0$ is truncated after two terms ($F_R \approx c_0 + c_1 F_A$), it predicts how much F_R for $F_A = 1$ Newton? 2 Newtons? 10^5 Newtons? Can fracture happen?

$F_R = -1$Newton, -2Newtons, -10^5Newtons. This system cannot fracture, as $F_R = -F_A$.

3.4 Breaking Point

What is the minimum number of terms in the Taylor series that must be retained to make F_R do something other than double when F_A is doubled?

At least one nonlinear term must be retained.

3.5 Extremes

Consider the response force F_R at positive and negative F_A. Is it approximately an even or odd function? If this were exactly true (i.e., symmetry or antisymmetry between pushing and pulling) what is the first term in the Taylor series that could make F_R sensitive to amplitude of F_A? It is an odd function. Thus, we expect the $c_3(F_A)^3$ term to be the first amplitude sensitive one.

4 Incroyable!

Indicate why the following equations cannot be true.

4.1 Q1

$$e^{i\theta} + e^{-i\theta} = \sqrt{(2-3)(2+3)}$$

LHS is pure real. RHS is pure imaginary.

4.2 Q2

$$\mathbf{C} \times \mathbf{A} \cdot \mathbf{B} = \mathbf{D}, \qquad \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \text{ are vectors}$$

LHS doesn't make sense, since the input to curl must be a vector or pseudovector.

4.3 Q3

$$\oint_{\partial A} \nabla\phi \cdot \mathrm{d}\mathbf{l} = 1, \qquad \text{(Integral over } \partial A \text{ follows a closed loop enclosing an area).}$$

By Stokes/Curl Theorem, $0 = \iint_A (\nabla \times \nabla\phi) \cdot \hat{\mathbf{n}}\, \mathrm{d}A = \oint_{\partial A} \nabla\phi \cdot \mathrm{d}\mathbf{l}$.

4.4 Q4

From 1 to 10, what's your favorite color of the alphabet? (seen in GeoChem on a T-shirt!).

Dimensions!

4.5 Q5

$$\tan x \geq \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

Doh! I meant to write $\tanh x$, not $\tan x$, in which case the LHS would have been bounded between -1 and 1. In any case, the RHS does not converge, as it is the harmonic series minus 1.