# Fall 2014 GEOL0350–GeoMath Final Exam v1

## 1 ODElay

We have already found the general solution to an ordinary differential equation for seismic wave displacements $(x, y)$ away from their steady positions $(x^*, y^*)$ at a given location and it is (where $c_1, c_2, c_3$ may be complex constants):

$$\begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{(i-0.1)t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{(-i-0.1)t} + c_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-10t}.$$

### 1.1 One Behavior

If $(x - x^*, y - y^*) = (1, 1)$ and $\frac{\mathrm{d}(x-x^*,y-y^*)}{\mathrm{d}t} = (-10, -10)$ at $t = 0$, what happens at larger $t$?

As this initial condition requires $c_1 = c_2 = 0$ and $c_3 = 1$, the solution rapidly decays back to equilibrium.

### 1.2 Another Behavior

What if $(x - x^*, y - y^*) = (-2, 2)$ at $t = 0$, what kinds of motion result?

In this case, the motion will oscillate as it decays back to equilibrium.

### 1.3 Quandary

How does one distinguish $c_1$ from $c_2$ since both describe initial positions along (1,-1)?

This problem will require two initial conditions, e.g., one on initial displacement and one on rate of change of displacement to set both coefficients.

### 1.4 Fixed Point

What is the long time behavior of this system, regardless of initial conditions?

All eigenvalues have a negative real part, so the fixed point is stable and return to equilibrium is expected.

### 1.5 System

What category of ordinary differential equation would you guess was solved by this solution?

It would be linear, constant-coefficient and third order.

# 2 PDQ PDE

Consider two functions that satisfy the two-dimensional Poisson equation and two that satisfy Helmholtz's equation within the two-dimensional domain $-1 \leq x \leq 1, -1 \leq y \leq 1$:

$$\nabla^2\phi = f(x,y), \qquad \nabla^2\psi = f(x,y),$$
$$\nabla^2\alpha + k^2\alpha = 0, \qquad \nabla^2\beta + k^2\beta = 0.$$

## 2.1 Super 1

Use superposition to determine an equation that $\phi - \psi$ solves. It solves Laplace's equation:

$$\nabla^2\phi - \nabla^2\psi = \nabla^2(\phi - \psi) = 0.$$

## 2.2 Super 2

Now determine an equation that $\alpha - \beta$ solves. Helmholtz's equation:

$$\nabla^2\alpha + k^2\alpha - (\nabla^2\beta + k^2\beta) = \nabla^2(\alpha - \beta) + k^2(\alpha - \beta) = 0$$

## 2.3 BCs

Suppose $\phi$ and $\psi$ both satisfy the following boundary conditions: $\psi = \phi = 0$ on $x = \pm 1$, and $\psi = \phi = 1$ on $y = \pm 1$ in addition to their field equations. Now what equations does $\phi - \psi$ solve, and what is the solution for $\phi - \psi$?

Field equation is still $\nabla^2\phi - \nabla^2\psi = \nabla^2(\phi - \psi) = 0$, but the boundary condition is $\phi - \psi = 0$ on all boundaries. Thus, since Laplace's equation prevents interior anomalies there is only one solution, $\phi - \psi = 0$, which means $\phi = \psi$.

## 2.4 BCs 2

Suppose $\phi$ and $\psi$ both satisfy the following boundary conditions: $\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial x} = 0$ on $x = \pm 1$, and $\frac{\partial\phi}{\partial y} = \frac{\partial\psi}{\partial y} = 0$ on $y = \pm 1$ in addition to their field equations. Now what equations does $\phi - \psi$ solve, and what is the solution?

Field equation is still $\nabla^2\phi - \nabla^2\psi = \nabla^2(\phi - \psi) = 0$, but the boundary condition is $\frac{\partial\phi - \psi}{\partial x} = 0$ on all boundaries. Thus, since Laplace's equation prevents interior anomalies only constant solutions are allowed, $\phi - \psi =$constant

# 3 Absolut Dynamical Systems

Consider the following dynamical system, which is extremely nonlinear but only at one point.

$$\dot{x} = r - |x| \tag{1}$$

## 3.1 Sketch $\dot{x}$ versus $x$ for $r < 0$, $r = 0$, $r > 0$ and denote stable and unstable fixed points.

## 3.2 Sketch $x^*$ versus $r$.

## 3.3 What kind of bifurcation is exhibited by (1)?

bluesky

## 3.4 Contrast the bifurcation in system (1) versus the one in (2).

$$\dot{x} = r(1 - |x|) \tag{2}$$

# 4 Stats–Whole Lotta Shakin

The Gutenberg-Richter Law gives the number $N$ of earthquakes in a region of time period of at least magnitude $M$. It can be expressed under simple conditions over a time window and location where there are 100 earthquakes of any magnitude as:

$$N = 10^{2-M}$$

A useful hint in this question will be:

$$\int^x 10^{2-M}\,\mathrm{d}M = -\frac{10^{2-x}}{\log 10}$$

Also note that $M$ ranges from 0 to $\infty$.

## 4.1 Normalization

The probability density function $\rho(M)$ is proportional to the number of events in this case, but is normalized differently, so $\rho(M) = C10^{2-M}$. Use the integral formula above to find $C$ and express the probability density function for the Gutenberg-Richter law.

$$1 = \int_0^\infty C10^{2-M}\,\mathrm{d}M = -C\frac{10^{2-\infty}}{\log 10} + C\frac{10^{2-0}}{\log 10} = C\frac{10^{2-0}}{\log 10} \to C = \frac{\log 10}{100}$$

## 4.2 Likelihood

How likely is a magnitude 5 or greater event versus a magnitude 4 or greater event?

$$\int_5^\infty \frac{\log 10}{100} 10^{2-M}\,\mathrm{d}M = \frac{1}{100,000}, \qquad \int_4^\infty \frac{\log 10}{100} 10^{2-M}\,\mathrm{d}M = \frac{1}{10,000}$$

Ten times less likely.

## 4.3 Percentile

What magnitude is the 99th percentile?

$$\int_2^\infty \frac{\log 10}{100} 10^{2-M}\,\mathrm{d}M = \frac{1}{100}$$

Thus, magnitude 2.0 is the 99th percentile, as 99/100 events occur with smaller magnitude.

## 4.4 Hypothesis

With significance $p < 0.01$, if only one earthquake was observed, what magnitude or greater would be needed to reject the hypothesis that this Gutenberg-Richter Law applies?

Any earthquake over magnitude 2.0, as this is the 99th percentile.

# 5 Stats–Central Limit & Monte Carlo

A new mode of climate variability is detected with a timescale shorter than a month, and it involves three different locations of the atmospheric jet stream. They are not of equal likelihood, but they are equally far from one another in distance based on a unit $L$. By looking back at historical records, it is determined that $y = 1L$ occurs 250 out of 2500 months, $y = 2L$ occurs 500 out of 2500 months, $y = 3L$ occurs 750 out of 2500 months, and $y = 4L$ occurs 1000 out of 2500 months. No other locations are observed.

## 5.1 Draw

Draw a histogram of the historical record data.

## 5.2 Mean

What is the average position $y$ of the jet stream?

$$
\begin{aligned}
\langle x\rangle &= L(250*1+500*2+750*3+1000*4)/2500 \\
&= L(1+2*2+3*3+4*4)/10 = (1+4+9+16)/10 = 3L
\end{aligned}
$$

## 5.3 Variance

What is the variance of the position?

$$
\begin{aligned}
\sigma^2 &= L^2(250*(1-3)^2+500*(2-3)^2+750*(3-3)^2+1000*(4-3)^2)/2500 \\
&= L^2(4+2*1+0+4*1)/10 = 10/10 = L^2
\end{aligned}
$$

## 5.4 Combine 2

If 2 independent months are chosen, what is the likelihood that at least 1 of them will have a jet stream at $y = L$?

$$p(A + B) = p(A) + p(B) - p(AB) = 1/10 + 1/10 - 1/100 = 19/100.$$

## 5.5 Mean of 4

If 4 independent months are chosen, use the central limit theorem to estimate the likelihood that the mean of these 4 months will be lower than $2L$.

Even though this distribution is not Gaussian, the mean will become close to a Gaussian distribution.

$$\sigma_4 = L/\sqrt{4}.$$

2 is $2\sigma$ below the mean, so 95.4% of the values are outside. One-tailed estimate yields 2.3% below this value, or, noting the odd shape of the distribution we might choose two-tailed 4.6%...

## 5.6 Bootstrapping

How does one use bootstrapping to improve this estimate of a particular 4 month sample? What about uncertainties on other statistics of the distribution not subject to the central limit theorem (e.g., skewness, kurtosis)?

It is clear that a Gaussian cannot be achieved with such small numbers of samples from the two-tailed vs. one-tailed test. Bootstrapping would help provide a better estimate of the pdf of 4-samples-at-a-time statistics on mean, variance, and pdf. The method would be to take the 4 mo. and randomly resample from this set of 4, calculating all statistics on each synthetic set. The histogram of all such synthetic data can be used to estimate pdfs.

## 5.7 Monte Carlo

Since we have a larger set of data than just 4 months from the historical data, how might Monte Carlo methods be used to make an even better estimate of the statistics of *any* 4 month's mean and uncertainty in *any* 4 month's samples of skewness and kurtosis drawn from the historical distribution?

We could repeatedly draw 4 month samples from the distribution plotted in the histogram, calculating the statistics on each set of 4. We could build the pdf of each.