Fall 2019 GEOL0350–GeoMath Final Exam

1 Gotta Catch 'Em All

Consider the following nonlinear ODE:

$$\frac{dx}{dt} = -x\left(x^2 + c^2 - 1\right), \tag{1}$$

Where c and R are real parameters.

We will examine the phase space and steady solutions of this equation.

1.1 Steady States

Plot the steady state solutions to this problem as the parameter c varies from -2 to 2–i.e., the bifurcation diagram.

Setting $dx/dt = 0$, we find 1 or 3 solutions.

$$0 = -x(x^2 + c^2 - 1), \tag{2}$$

$$0 = x, \tag{3}$$

$$x^2 + c^2 = 1., \text{ i.e., a circle of radius 1} \tag{4}$$

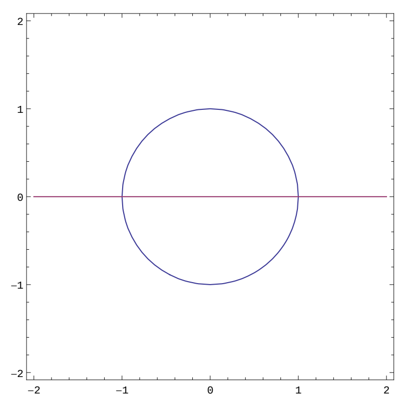

Figure 1: Bifurcation

1.2 Phase Space

Plot the phase space diagrams (dx/dt vs x) for two different cases: $c = 0$ and $c = 1$.

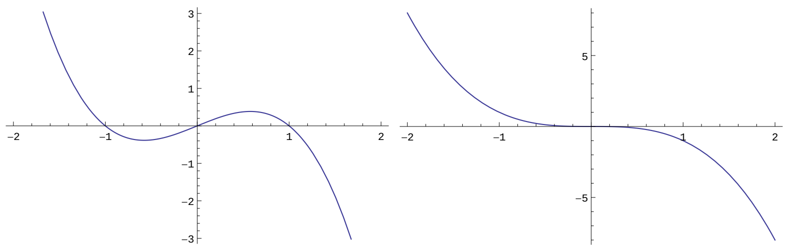

Figure 2: Phase Space

1.3 Stable or unstable?

Use the results from 1.2 (Phase Space) to determine which branches of (1.1) are stable and unstable.

The $x = 0$ branch is unique and stable when $|c| \geq 1$. When $|c| \leq 1$, the upper and lower branches are stable and the middle solution is unstable.

1.4 Local linearization

Linearize (by performing a Taylor series) the equation about each of the solutions ($x = x^*$) when $c = 0$ and $c = 1$, and then decide if the each of these solutions are stable or unstable when $c = 0$ and $c = 1$.

$$c = 0, x = 0 : \frac{dx}{dt} = x \qquad \text{unstable,} \tag{5}$$

$$c = 0, x = 1 : \frac{d(x-1)}{dt} = -2(x-1) \qquad \text{stable,} \tag{6}$$

$$c = 0, x = -1 : \frac{d(x+1)}{dt} = -2(x+1) \qquad \text{stable,} \tag{7}$$

$$c = 1, x = 0 : \frac{dx}{dt} = 0 \qquad \text{linearly neutral, nonlinearly stable} \tag{8}$$

You can use the coefficient of the leading Taylor series term to estimate linear stability. In the last case, this zero–implying neutrality, but the third order term actually implies nonlinear stability.

2 Xtreme PDF!

Three different probability distribution functions are being considered to explain the results of an experiment to measure x, which is always positive. They are:

$$\rho_l(x;\sigma) = \frac{1}{x\sigma\sqrt{2\pi}}e^{-(\ln[x])^2/(2\sigma^2)}, \quad \rho_e(x;\sigma) = \frac{\sqrt{2}}{\sigma}e^{-\sqrt{2}x/\sigma}, \quad \rho_n(x;\sigma) = \frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-(x)^2/(2\sigma^2)} \tag{9}$$

Hints:

$$\int_0^\infty \rho_l(x;\sigma)\,\mathrm{d}x = 1.0, \quad \int_0^\infty \rho_e(x;\sigma)\,\mathrm{d}x = 1.0, \quad \int_0^\infty \rho_n(x;\sigma,\mu)\,\mathrm{d}x = 1.0 \tag{10}$$

$$\int_\sigma^\infty \rho_l(x;\sigma)\,\mathrm{d}x = 0.5, \quad \int_\sigma^\infty \rho_e(x;\sigma)\,\mathrm{d}x \approx 0.243, \quad \int_\sigma^\infty \rho_n(x;\sigma)\,\mathrm{d}x \approx 0.317, \tag{11}$$

$$\int_{2\sigma}^\infty \rho_l(x;\sigma)\,\mathrm{d}x = 0.244, \quad \int_{2\sigma}^\infty \rho_e(x;\sigma)\,\mathrm{d}x \approx 0.059, \quad \int_{2\sigma}^\infty \rho_n(x;\sigma)\,\mathrm{d}x \approx 0.045, \tag{12}$$

$$\int_{3\sigma}^\infty \rho_l(x;\sigma)\,\mathrm{d}x = 0.136, \quad \int_{3\sigma}^\infty \rho_e(x;\sigma)\,\mathrm{d}x \approx 0.014, \quad \int_{3\sigma}^\infty \rho_n(x;\sigma)\,\mathrm{d}x \approx 0.002, \tag{13}$$

$$\tag{14}$$

2.1 Extreme

Which of the probability distributions has the greatest likelihood to measure $x > \sigma$?

$\rho_l(x;\sigma)$

2.2 Hyperextreme

Which of the probability distributions has the greatest likelihood of finding values where $x > 2\sigma$?

$\rho_l(x;\sigma)$

2.3 Plot

Plot these three distributions. You may use a log ρ axis or not as you please.

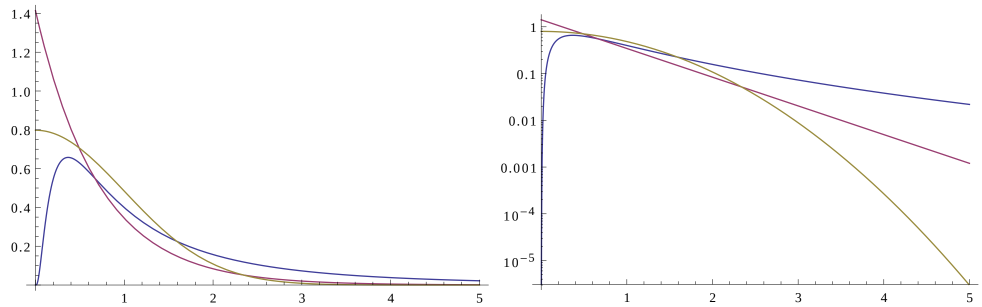

Figure 3: PDFs

2.4 Intermittent

How can you, by making just histograms of varying bin sizes and using logarithms, tell which of these distributions is most appropriate.

If the histogram is $h(x)$, plot $\ln(h(x))$ and $\ln(h(\ln(x))$. If the former is a quadratic or a line, then the fitted distribution is the normal or exponential fit, respectively. If the pdf of the $\ln(x)$ is quadratic, then it is the lognormal distribution.

3 Superfishy

Consider the partial differential equation for unknown fields ϕ, Ψ:

$$\nabla \cdot (\nabla \phi + \nabla \times \Psi) = 0 \quad (15)$$

3.1 What kind

What kind of equation is this? Partial? Linear, homogeneous, nonlinear? Etc?

It is a second-order, linear, homogeneous, partial differential equation.

3.2 How are they coupled?

Describe how different solutions for Ψ affect the solutions for ϕ.

Not at all, the value of ϕ does not survive the differentiation by the divergence, and so any value of Ψ leaves ϕ unaffected.

3.3 Divide and Conquer

Use separation of variables to write the general solution to this equation for ϕ in a 2D square domain.

Given the fact that Ψ doesn't contribute, this is just the Poisson equation for ϕ. Assume $\psi = X(x)Y(y)$, then $X''/X = k^2 = -Y''/Y$. Thus, $X = e^{kx}$, $y = e^{iky}$. The general solution is $\psi = \sum_k A_k e^{kx} e^{iky}$.

3.4 Solve

Suppose $\phi = 1$ and $\Psi = 1$ on all boundaries of a rectangular 2D domain. What is the solution for ϕ?

$\phi = 1$, which is easily found by noting that due to Fourier's trick and the orthogonality of Fourier series, only $k = 0$ can be matched to a steady boundary condition. $k = 0$ implies the solution is a constant, and that constant is 1.

3.5 Solve

Suppose $\phi = 1$ and $\Psi = 1$ on all boundaries of a rectangular 2D domain. What is the solution for Ψ?

Indeterminate. More information is needed.

4 Fishy

Consider the Poisson and the Helmholtz equations:

$$\nabla^2 \phi_p = 0, \tag{16}$$

$$\nabla^2 \phi_h + k^2 \phi_h = 0, \tag{17}$$

in the special case where k is a very small number and ϕ_p and ϕ_k have the same boundary conditions. These equations are obviously similar in this limit, but how?

4.1 Amplitude dependence

Suppose we choose boundary conditions that make ϕ_p and ϕ_h very large in magnitude. Is the effect of k greater or smaller in affecting the outcome than if ϕ_p and ϕ_h are more modest in magnitude?

As both equations are linear, there is no effect from having different magnitudes of ϕ on their shape (i.e., superposition). Linear equations have solutions shapes that are amplitude-independent.

4.2 Smooth

How might one describe the "smoothness" or "wiggliness" of ϕ_p and ϕ_k?

By considering their Fourier transform. The larger their higher wavenumber coefficients relatively, the more wiggly they are. The smaller their higher coefficients, the more smooth they are.

4.3 Compare k to ...

Will the effect of k be more pronounced if the boundary conditions are such to produce smooth or wiggly ϕ functions?

The smoother the ϕ functions, the larger the k term will be, relatively. So, for very wiggly ϕ (high wavenumber), the k term will be negligible and the Helmholtz and Poisson solutions will be very similar. $\nabla^2\phi \gg k^2\phi$ when the solution is "wiggly", and $\nabla^2\phi \ll k^2\phi$ when it is smooth.

4.4 Exponential and Oscillatory

In class, we discussed how solutions to the Poisson equation are built up from a basis of functions that are oscillatory in one direction and exponentially decaying in the other. Contrasting versus solutions to the Helmholtz equation that are also oscillating and decaying, how are do the rates of oscillation and decay differ?

In the Poisson case, the oscillation wavenumber equals the decay rate. In the Helmholtz system the decay rate squared *plus* k^2 equals the oscillation rate. Thus, the oscillations are faster (more wiggly, smaller scale) for the same decay rate in the Helmholtz system versus the Poisson.