Fall 2019 GEOL0350–Homework 1
Based on Chapter 1 of the Notes (Series)

1 Limits of Series

1.1

Find the limit of

$$0.9 + 0.09 + 0.009 + 0.0009 + \ldots \tag{1}$$

$$S = a + ar + ar^2 + ar^3 + \ldots, \tag{2}$$
$$a = 0.9, \tag{3}$$
$$r = 0.1, \tag{4}$$
$$S = \frac{a}{1-r} = \frac{0.9}{1-0.1} = 1 \tag{5}$$

1.2

Find the limit of

$$1 + 1/3 + 1/9 + 1/27 + \ldots \tag{6}$$

$$S = a + ar + ar^2 + ar^3 + \ldots, \tag{7}$$
$$a = 1, \tag{8}$$
$$r = \frac{1}{3}, \tag{9}$$
$$S = \frac{a}{1-r} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2} \tag{10}$$

1.3

Find the limit of

$$0.1 + 0.1 + 0.1 + \dots \tag{11}$$

Formally, this series does not converge so it has no limit. Informally, we can see that it approaches ∞, but that is not a real, formal limit as you cannot ever say that you are within a finite distance of ∞.

1.4

Find the limit of

$$0.1 - 0.1 + 0.1 - 0.1 + \dots \tag{12}$$

Formally, this series does not converge so it has no limit. Informally, we can see that it oscillates about 0.05, but that is not the limit as a limit requires that it eventually becomes arbitrarily close to 0.05 (i.e., subsequent terms must get smaller and smaller).

2 Coriolis

Exercise 1.5 from the notes:
http://www.geo.brown.edu/research/Fox-Kemper/classes/GEOL0350_19/notes/0350notes.pdf.

An important parameter in the consideration of the physics of the rotating Earth is the Coriolis parameter: $2\Omega\sin(\varphi)$, where Ω is the angular rate of rotation of the earth in radians (2π in a day, or $2\pi/(24hr/s\cdot 3600s)$) and φ is the latitude. Taylor expand the first terms in this parameter around 30 degrees North, or $30°\cdot(2\pi\text{radians}/360°)=\pi/6$ radians. Carry out the expansion to the term including $(\varphi-\pi/6)^3$.

$$\sin(\varphi)=\sum_{n=0}^{\infty}\frac{(\varphi-\frac{\pi}{6})^n}{n!}\sin(\varphi)^{(n)}\left(\frac{\pi}{6}\right), \tag{13}$$

$$=\frac{(\varphi-\frac{\pi}{6})^0}{0!}\sin(\varphi)^{(0)}\left(\frac{\pi}{6}\right)+\frac{(\varphi-\frac{\pi}{6})^1}{1!}\sin(\varphi)^{(1)}\left(\frac{\pi}{6}\right)+\frac{(\varphi-\frac{\pi}{6})^2}{2!}\sin(\varphi)^{(2)}\left(\frac{\pi}{6}\right)+\dots,$$

$$=\sin\left(\frac{\pi}{6}\right)+(\varphi-\frac{\pi}{6})\cos\left(\frac{\pi}{6}\right)-\frac{(\varphi-\frac{\pi}{6})^2}{2}\sin\left(\frac{\pi}{6}\right)-\frac{(\varphi-\frac{\pi}{6})^3}{6}\cos\left(\frac{\pi}{6}\right)+\dots, \tag{14}$$

$$=\frac{1}{2}+(\varphi-\frac{\pi}{6})\frac{\sqrt{3}}{2}-\frac{(\varphi-\frac{\pi}{6})^2}{4}-\frac{(\varphi-\frac{\pi}{6})^3\sqrt{3}}{12}+\dots, \tag{15}$$

$$2\Omega\sin(\varphi)=\Omega+\Omega(\varphi-\frac{\pi}{6})\sqrt{3}-\Omega\frac{(\varphi-\frac{\pi}{6})^2}{2}-\Omega\frac{(\varphi-\frac{\pi}{6})^3\sqrt{3}}{6}+\dots \tag{16}$$

3 Taylor Rhymes with Baylor

Exercise 1.7 from the notes:
http://www.geo.brown.edu/research/Fox-Kemper/classes/GEOL0350_19/notes/0350notes.pdf.

Suppose the function $h(x)$ plotted in the figure is found by measuring topography along the x direction, and in particular consider fitting it with a series expansion near the point marked A. Sea level is $h=0$, so we are particularly interested in $h(x)=0$, which indicates the location of coastlines. The function is not known, but we consider the possibility of approximating a Taylor series expansion to it.

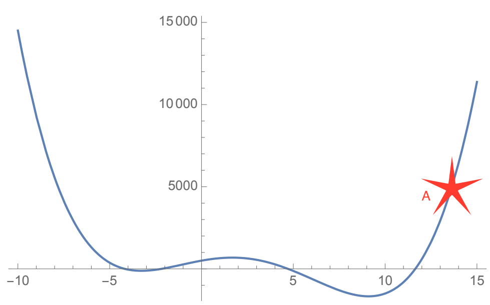

Figure 1: A function to be fit by Taylor expansion near point A at the star.

a, Counting: How many coastlines are there, that is how many solutions to $h(x)=0$ are there?

b, Constant: If the function is fit with a Taylor series and truncated at the first (constant) term, and if the truncated approximation is denoted $\tilde{h}_0(x)$, then how many solutions are there to $\tilde{h}_0(x) = 0$?
c, Variations: If the Taylor series is instead truncated after two terms ($\tilde{h}_1(x)$) then how many solutions are there to $\tilde{h}_1(x) = 0$, and what is the approximate value of the solution x?
d, How many?: What is the minimum number of terms in the Taylor series that must be retained to approximate all of the coastlines in the real function $h(x)$? Why?
e, Extremes: Consider the function at large magnitudes of positive and negative x. If the truncated Taylor series matches this behavior at large $|x|$, predict the sign of the coefficient in the largest power of x and whether the power is even or odd.

a: 4

b: None.

c: One, near $x = 12$.

d: Since there are 4 roots to the equation $h(x) = 0$, $\tilde{h}(x)$ must be at least a fourth-order polynomial to have this many roots, which has 5 terms of the Taylor series retained.

e: Positive and even power. Negative would tend toward $-\infty$, and even power since the function increases on both sides of A or 0.

4 Who is the Oddest?

Exercise 1.8 from the notes:
`http://www.geo.brown.edu/research/Fox-Kemper/classes/GEOL0350_19/notes/0350notes.pdf`.

Sometimes it is said that sin is an *odd function* and cos is an *even function*. Examine the sin and cos functions in (1.23) and explain what this means in terms of the exponents of x. Consider $\sin(x)$ versus $\sin(-x)$ and $\cos(x)$ versus $\cos(-x)$, how do they compare? How do odd and even functions compare under the sign reversal of their argument (i.e., the input to the function, x or $-x$) in general?

The powers of all of the terms in the Taylor series (1.23) are either all odd (in the case of sine) or all even (in the case of cosine). As odd powers reverse sign under reversal of the argument, e.g., $(-x)^3 = -x^3$, $\sin(-x) = -\sin(x)$. By contrast e.g., $(-x)^4 = x^4$, so $\cos(-x) = \cos(x)$. Also, since taking the derivative lowers the exponent of each term by one, it makes sense that the derivative of an odd function is even and vice versa.

5 Biggie Smalls

Exercise 1.9 from the notes:
`http://www.geo.brown.edu/research/Fox-Kemper/classes/GEOL0350_19/notes/0350notes.pdf`.

Examine each factor in the product that makes up (**??**). What makes them large or small as n increases?

It is important to think a bit about what leads the Taylor series to converge. The $n!$ in the denominator is a strong pull toward smaller and smaller numbers. In the examples (1.23) to (1.29), this factorial remains in many of the series. However, keeping $(x-a)$ small is also important–that is, the approximation is better the closer to the location where the derivatives are evaluated. Finally, the derivatives themselves need to not get larger and larger with repeated differentiation. In series where this occurs, such as (1.29), the factorial is cancelled out by the increasingly sharp derivative functions.
The factorial always decreases the size with increasing n. The $(x-a)^n$ can increase or decrease the size, depending on whether x is close to a. The derivative $f^{(n)}(a)$ tends to be noisier & larger with increasing n (as integrals tend to be smoother than the original signal, hence averaging). Examining the finite difference approximation for the following combination,

$$\frac{(x-a)^n \Delta^n f}{\Delta x^n}, \tag{17}$$

implies that "near" and "far" for $(x-a)$ is measured in terms of how far in x you need to go to make Δf sizeable or make f "wiggle" appreciably. The units of f provide the units of the whole Taylor approximation, while the units of $x-a$ need to match the units of Δx.