Fall 2019 GEOL0350–Homework 3
Based on Chapter 4 & 5 of the Notes (Dimensions, Partials)

1 Most Def

What is the difference between a parameter, a unit, and a dimension? A parameter is a measurable quantity that describes an aspect of a problem; a parameter can be dimensional or dimensionless. A unit is a dimensional quantity used for measuring other quantities sharing the same dimension by comparison (by ratio). For example, the size of a feature is a possible choice of unit, as is the meter. A dimension is a fundamental measure of space or time, such as length, time, or mass–the critical aspect of a dimension is that measurements of quantities with matching dimension can be related, perhaps after a conversion to matching units (e.g., we can compare lengths to lengths).

2 The Dims are Out!

Show that Ro, Re, and Ra are dimensionless by expanding their parameters into their dimensions.

$$[\mathrm{Ro}] = \left[\frac{U}{fL}\right] = \frac{\frac{L}{T}}{\frac{1}{T}L} = 1,$$

$$[\mathrm{Re}] = \left[\frac{UL}{\nu}\right] = \frac{\frac{L}{T}L}{\frac{L^2}{T}} = 1,$$

$$[\mathrm{Ra}] = [\mathrm{Gr}]\,[\mathrm{Pr}] = \left[\frac{g\Delta\rho L^3}{\nu^2\rho_0}\right]\left[\frac{\nu}{\kappa}\right] = \frac{\frac{L}{T^2}\frac{M}{L^3}L^3}{\frac{L^4}{T^2}\frac{M}{L^3}}\frac{\frac{L^2}{T}}{\frac{L^2}{T}} = 1 \times 1.$$

3 Run it up the flagpole

(From 2.36 of Wilcox, 1997)
Because of a phenomenon called vortex shedding, a flagpole will oscillate at a frequency ω when the wind blows at velocity U. The diameter of the flagpole is D and the kinematic viscosity of air is ν. Using dimensional analysis, develop an equation for ω as the product of a quantity independent of ν with the dimensions of ω and a function of all relevant dimensionless groupings. We have the following dimensional parameters and results,

$$[\omega] = \frac{1}{T}, \qquad [U] = \frac{L}{T}, \qquad [D] = L, \qquad [\nu] = \frac{L^2}{T},$$

Thus, Buckingham's Pi Theorem predicts $4-2=2$ independent dimensionless parameters. We choose the first as the Reynolds number of the pole. The second needs to incorporate the frequency, and we see that the Strouhal number works well for this parameter.

$$\mathrm{Re}=\frac{UD}{\nu}, \qquad \mathrm{St}=\frac{\omega D}{U}$$

Other choices would be acceptable–in particular any power of these two ratios! If there is only one dimensionless parameter, it must be constant. If there are two, then we propose that a function relates them. If we make this assumption, and then solve for the frequency,

$$\frac{\omega D}{U}=\mathrm{St}=f(\mathrm{Re})=f\left(\frac{UD}{\nu}\right)\longrightarrow\omega=\frac{U}{D}f(\mathrm{Re}).$$

Quod erat inveniendum.

4 Gallia est omnis divisa in partes tres

Explain why you think the terms "partial" derivative and "total" derivative apply as they do. Partial derivatives describe only part of the change in a dependent variable with respect to a set of independent variables–in fact just the change associated with one independent variable at a time. Total derivatives describe the change associated with all of the independent variables in turn.

5 Multi-platinum Taylor

Problems 4.2.4 of Boas (2006). Use the multivariate Taylor series to evaluate the expansion of e^{xy}.

4.2.4:

$$f(x,y)=e^{xy},$$

$$\left.\frac{\partial e^{xy}}{\partial x}\right|_{x=y=0}=\left.ye^{xy}\right|_{x=y=0}=0, \qquad \left.\frac{\partial^n e^{xy}}{\partial x^n}\right|_{x=y=0}=\left.y^n e^{xy}\right|_{x=y=0}=0,$$

$$\left.\frac{\partial e^{xy}}{\partial y}\right|_{x=y=0}=\left.xe^{xy}\right|_{x=y=0}=0, \qquad \left.\frac{\partial^n e^{xy}}{\partial y^n}\right|_{x=y=0}=\left.x^n e^{xy}\right|_{x=y=0}=0.$$

Therefore, we only need to worry about mixed derivatives.

$$\left.\frac{\partial^2 e^{xy}}{\partial x\partial y}\right|_{x=y=0}=\left.\frac{\partial\left(ye^{xy}\right)}{\partial y}\right|_{x=y=0}=\left.e^{xy}+xye^{xy}\right|_{x=y=0}=1,$$

$$\left.\frac{\partial^3 e^{xy}}{\partial x^2\partial y}\right|_{x=y=0}=\left.\frac{\partial\left(y^2e^{xy}\right)}{\partial y}\right|_{x=y=0}=\left.2ye^{xy}+xy^2e^{xy}\right|_{x=y=0}=0,$$

$$\left.\frac{\partial^4 e^{xy}}{\partial x^2\partial y^2}\right|_{x=y=0}=\left.\frac{\partial^2\left(y^2e^{xy}\right)}{\partial y^2}\right|_{x=y=0}=\left.2e^{xy}+2xye^{xy}+2xye^{xy}+x^2y^2e^{xy}\right|_{x=y=0}=2.$$

In fact, we only need to worry about mixed derivatives where the number of x-derivatives equals the number of y derivatives, or only the even differential orders. The value of the $\frac{\partial^{2n} f(x,y)}{\partial x^n \partial y^n}$ derivative will be $n!$. We can use the factorial form of the binomial coefficient to expand, noting that only the matched derivatives survive,

$$\begin{aligned} f(x,y) = e^{xy} &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{[x-a]^{n-k}\,[y-b]^{k}}{k!(n-k)!} \left(\left. \frac{\partial^n e^{xy}}{\partial x^{n-k} \partial y^k} \right|_{x=a,y=b} \right), \\ &= \sum_{n=0}^{\infty} \frac{x^n y^n}{n! n!} n! \\ &= \sum_{n=0}^{\infty} \frac{x^n y^n}{n!} \end{aligned}$$

Which is the same form as the expansion for e^z with $z = xy$.

6 Error Propagation

Problem 4.4.8 of Boas (2006). About how much (in percent) does an error of 1% in a and b affect $a^2 b^3$?

We can do this by brute force,

$$\begin{aligned} b &\to b + \Delta b, \qquad a \to a + \Delta a, \\ a^2 b^3 &\to (a+\Delta a)^2 (b+\Delta b)^3 \\ &= a^2b^3 + 2ab^3(\Delta a) + b^3(\Delta a)^2 + 3a^2b^2(\Delta b) + 6ab^2(\Delta a)(\Delta b) + 3b^2(\Delta a)^2(\Delta b) + 3a^2b(\Delta b)^2 + 6ab(\Delta a)(\Delta b)^2 \\ &+ 3b(\Delta a)^2(\Delta b)^2 + a^2(\Delta b)^3 + 2a(\Delta a)(\Delta b)^3 + (\Delta a)^2(\Delta b)^3. \end{aligned}$$

But, since $(\Delta a)^2 \ll \Delta a$, and $(\Delta b)^2 \ll \Delta b$, we can neglect all of the higher powers of the deviations.

$$\begin{aligned} b &\to b + \Delta b, \qquad a \to a + \Delta a, \\ a^2 b^3 &\to (a+\Delta a)^2 (b+\Delta b)^3 \\ &\approx a^2b^3 + 2a^2b^3 \frac{\Delta a}{a} + 3a^2b^3 \frac{\Delta b}{b} \end{aligned}$$

Which predicts (because of the coefficients) that a 1% error in a will lead to a 2% error in a^2b^3, and a 1% error in b will lead to a 3% error in a^2b^3. If the errors in the variables are random and uncorrelated, we'd expect the combined effect to be $\sqrt{2\%^2 + 3\%^2} \approx 3.6\%$.

To do this much more rapidly using calculus, we take the natural logarithm and differentiate:

$$\begin{aligned} \ln\left(a^2b^3\right) &= 2\ln a + 3\ln b, \\ \frac{\mathrm{d}\left(a^2b^3\right)}{a^2b^3} &= 2\frac{\mathrm{d}a}{a} + 3\frac{\mathrm{d}b}{b}. \end{aligned}$$

Which is equivalent to the same result. You can also just differentiate and the collect terms without using the logarithm, but extra steps are needed.

7 Just Work It Out

Problem 4.1.1 of Boas (2006). If $u = \frac{x^2}{x^2+y^2}$, find the partial derivatives of u with respect to x and y.

$$\begin{aligned}
u &= \frac{x^2}{x^2+y^2}, \\
\frac{\partial u}{\partial x} &= \frac{2x}{x^2+y^2} - \frac{x^2 \times 2x}{(x^2+y^2)^2} = \frac{2xy^2}{(x^2+y^2)^2}, \\
\frac{\partial u}{\partial y} &= -\frac{2yx^2}{(x^2+y^2)^2}.
\end{aligned}$$

8 Streamfunction and Velocity Potential

As we come to understand how fluids move, we will be interested in breaking up the velocity field into a part that results from a convergence or divergence of flow and a part that results from a swirling or rotating flow. This can be done with the Helmholtz Decomposition, which we will address in a later chapter on vector analysis. For now, we'll just test some examples.

The relationship between velocity (u, v) and streamfunction (ψ) and velocity potential (ϕ), in 2D, is just

$$u = -\frac{\partial \psi}{\partial y} - \frac{\partial \phi}{\partial x}, \tag{1}$$

$$v = \frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}. \tag{2}$$

Both velocity components, as well as ψ and ϕ should be interpreted as spatial fields–that is, they are all functions of x and y and have a value at every point in space.

The Jacobian is a useful function for evaluating the advection by flow due to a streamfunction alone, which in 2D is just

$$J(A, B) = \begin{vmatrix} \frac{\partial A}{\partial x} & \frac{\partial A}{\partial y} \\ \frac{\partial B}{\partial x} & \frac{\partial B}{\partial y} \end{vmatrix} = \frac{\partial A}{\partial x}\frac{\partial B}{\partial y} - \frac{\partial A}{\partial y}\frac{\partial B}{\partial x}. \tag{3}$$

a) If $\phi = 0$, show that $J(\psi, f(x,y)) = u\frac{\partial f(x,y)}{\partial x} + v\frac{\partial f(x,y)}{\partial y}$.

b) If A is a function of B, rather than independently varying in x and y, show that $J(B, A(B)) = 0$.

$$(a) J(\psi, f(x,y)) = \frac{\partial \psi}{\partial x}\frac{\partial f(x,y)}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial f(x,y)}{\partial x} = u\frac{\partial f(x,y)}{\partial x} + v\frac{\partial f(x,y)}{\partial y}.$$

$$(b)\text{Chain rule: } J(B, A(B)) = \frac{\partial B}{\partial x}\frac{\partial A(B)}{\partial y} - \frac{\partial B}{\partial y}\frac{\partial A(B)}{\partial x} = \frac{\partial B}{\partial x}\frac{\mathrm{d}A(B)}{\mathrm{d}B}\frac{\partial B}{\partial y} - \frac{\partial B}{\partial y}\frac{\mathrm{d}A(B)}{\mathrm{d}B}\frac{\partial B}{\partial x} = 0.$$

9 Commuting to Work

Problem 4.12.11 of Boas (2006). Find

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{3-x}^{x^2}(x-t)\,\mathrm{d}t \tag{4}$$

a) by integrating first, and b) by differentiating first.

4.12.11: a) Integrating first,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\int_{3-x}^{x^2}(x-t)\,\mathrm{d}t &= \frac{\mathrm{d}}{\mathrm{d}x}\left[xt - \frac{1}{2}t^2\Big|_{t=3-x}^{t=x^2}\right],\\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left[x^3 - \frac{1}{2}x^4 - x(3-x) + \frac{1}{2}(3-x)^2\right] = \frac{\mathrm{d}}{\mathrm{d}x}\left[-\frac{1}{2}x^4 + \frac{3}{2}x^3 - 6x + \frac{9}{2}\right],\\
&= -2x^3 + 3x^2 + 3x - 6.
\end{aligned}$$

b) Differentiating first (using Leibniz's rule):

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\int_{3-x}^{x^2}(x-t)\,\mathrm{d}t &= (x-x^2)\frac{\mathrm{d}x^2}{\mathrm{d}x} - (x-(3-x))\frac{\mathrm{d}(3-x)}{\mathrm{d}x} + \int_{3-x}^{x^2}\frac{\mathrm{d}(x-t)}{\mathrm{d}x}\,\mathrm{d}t,\\
&= (x-x^2)2x + (x-(3-x)) + \int_{3-x}^{x^2}\mathrm{d}t,\\
&= (x-x^2)2x + (x-(3-x)) + x^2 - (3-x),\\
&= -2x^3 + 3x^2 + 3x - 6.
\end{aligned}$$

10 The D'Alembert Wave

Problem 4.7.23 of Boas (2006).
If $u = f(x-ct) + g(x+ct)$, show that $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$.

We can do this by direct integration, but it is more fun to note that both $f(x-ct)$ and $g(x+ct)$ solve the wave equation (they are D'Alembert's solutions),

$$\frac{\partial^2 f(x-ct)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 f(x-ct)}{\partial t^2} = \frac{\mathrm{d}^2 f(x-ct)}{\mathrm{d}(x-ct)^2} - \frac{c^2}{c^2}\frac{\mathrm{d}^2 f(x-ct)}{\mathrm{d}(x-ct)^2} = 0.$$
$$\frac{\partial^2 g(x+ct)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 g(x+ct)}{\partial t^2} = \frac{\mathrm{d}^2 g(x+ct)}{\mathrm{d}(x+ct)^2} - \frac{c^2}{c^2}\frac{\mathrm{d}^2 g(x+ct)}{\mathrm{d}(x+ct)^2} = 0.$$

The wave equation is linear and homogeneous, so if $f(x-ct)$ is a solution and $g(x+ct)$ is a solution, then so is their sum!

References

Boas, M. L., 2006. Mathematical methods in the physical sciences, 3rd Edition. Wiley, Hoboken, NJ.

Wilcox, D. C., 1997. Basic fluid mechanics. DCW Industries, La Cañada, Calif.