Fall 2019 GEOL0350–Homework 4
Based on Chapter 6 & 7 of the Notes (Integrals, Fourier)

1 Fundamentals I

Show that for a cube with edges of length one, integrating 3 over the volume is the same as area integrating $(\hat{\mathbf{x}}+\hat{\mathbf{y}}+\hat{\mathbf{z}})\cdot\hat{\mathbf{n}}$ over the six faces of the cube, where $\hat{\mathbf{n}}$ is the *outward normal unit* vector, which points outward perpendicularly from each face, and $\hat{\mathbf{x}}, \hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are the unit normal vectors in the x, y, z directions, respectively. (Hint: center the cube on the origin).

$$\iiint\limits_V 3\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iint\limits_S (x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) \cdot \mathbf{n}\, \mathrm{d}A \tag{1}$$

The volume of the cube is 1, so the integral over the volume is just 3. Each of the faces lies at $x = \pm\frac{1}{2}, y = \pm\frac{1}{2}, z = \pm\frac{1}{2}$, so the integrand is 1/2 for each face and the area of each face is 1, thus, the area integral over the six faces is 3.

2 Fundamentals II

Find

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x t^2\,\mathrm{d}t \tag{2}$$

a) by integrating first, and b) by differentiating first.

a)

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x t^2\,\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}x}\,\frac{1}{3}t^3\bigg|_0^x = \frac{\mathrm{d}}{\mathrm{d}x}\,\frac{1}{3}x^3 = x^2 \tag{3}$$

b) Using (5.9),

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x t^2\,\mathrm{d}t = x^2 \tag{4}$$

3 Fundamentals III

Find

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{x/2}^{x}(x-t)\,\mathrm{d}t \tag{5}$$

a) by integrating first, and b) by differentiating first.

a)

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{x/2}^{x}(x-t)\,\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}x}\left(xt - t^2/2\right)\Big|_{x/2}^{x} = \frac{\mathrm{d}}{\mathrm{d}x}\left((x^2 - x^2/2) - (x^2/2 - x^2/8)\right) = x/4 \tag{6}$$

b) Using (5.9),

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{x/2}^{x}(x-t)\,\mathrm{d}t = (x-x) - (x - x/2)/2 + \int_{x/2}^{x}\frac{\mathrm{d}}{\mathrm{d}x}(x-t)\,\mathrm{d}t = -x/4 + \int_{x/2}^{x}\mathrm{d}t = -x/4 + x/2 = x/4 \tag{7}$$

4 Volumes

Set up and evaluate integrals to calculate the volume of an $L_x \times L_y \times L_z$ rectangular solid, a cylinder of radius R and height h, and a sphere of radius R.

$$\begin{aligned}
V &= \int_0^{L_z}\int_0^{L_y}\int_0^{L_x}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = L_xL_yL_z,\\
V &= \int_0^{h}\int_0^{2\pi}\int_0^{R} r\,\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}z = \pi R^2 h,\\
V &= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{4}{3}\pi R^3.
\end{aligned}$$

5 Earth

Redo the cylinder and sphere volume calculations from the previous exercise, but with all integrals and integrands expressed *in earth coordinates*. Hint: It is easiest to consider the cylinder as sitting with its base on the origin (rather than centered on the origin). Then break up the integral into two parts. First, there is the conic section that extends from $\vartheta = \tan^{-1}(h/R)$ to $\vartheta = \pi/2$ and is bounded at the surface of the top of the cylinder. This surface can be described by $\mathfrak{z} = h/\sin\vartheta$. The other surface to consider is the outer shell of the cylinder. This surface can be described by the function $\mathfrak{z} = R/\cos\vartheta$, and it is relevant for $\vartheta = 0$ to $\vartheta = \tan^{-1}(h/R)$.

Rectangular Solid: The earth coordinates are based on a sphere of radius r_0, which is supposed to be the mean radius of the earth. We can make this problem a little cleaner by assuming that the radius of our cylinder is much larger than the r_0, and therefore just using $r_0 = 0$. It is also easiest to consider the cylinder as sitting with its base on the origin (rather than centered on the origin). We break up the integral into two parts. First, there is the conic section that extends from $\vartheta = \tan^{-1}(h/R)$ to $\vartheta = \pi/2$ and is bounded at the surface of the top of the cylinder. This surface can be described by $\mathfrak{z} = h/\sin\vartheta$. The other surface to consider is the outer shell of the cylinder. This surface can be described by the function $\mathfrak{z} = R/\cos\vartheta$, and it is relevant for $\vartheta = 0$ to $\vartheta = \tan^{-1}(h/R)$. Thus,

$$\begin{aligned}
V &= \int_0^{\tan^{-1}(h/R)} \int_0^{2\pi} \int_0^{R/\cos\vartheta} \mathfrak{z}^2 \cos\vartheta \, \mathrm{d}\mathfrak{z} \, \mathrm{d}\phi \, \mathrm{d}\vartheta + \int_{\tan^{-1}(h/R)}^{\pi/2} \int_0^{2\pi} \int_0^{h/\sin\vartheta} \mathfrak{z}^2 \cos\vartheta \, \mathrm{d}\mathfrak{z} \, \mathrm{d}\phi \, \mathrm{d}\vartheta, \\
&= 2\pi \int_0^{\tan^{-1}(h/R)} \int_0^{R/\cos\vartheta} \mathfrak{z}^2 \cos\vartheta \, \mathrm{d}\mathfrak{z} \, \mathrm{d}\vartheta + 2\pi \int_{\tan^{-1}(h/R)}^{\pi/2} \int_0^{h/\sin\vartheta} \mathfrak{z}^2 \cos\vartheta \, \mathrm{d}\mathfrak{z} \, \mathrm{d}\vartheta, \\
&= \frac{2\pi}{3} \int_0^{\tan^{-1}(h/R)} \frac{R^3}{\cos^2\vartheta} \, \mathrm{d}\vartheta + \frac{2\pi}{3} \int_{\tan^{-1}(h/R)}^{\pi/2} \frac{h^3 \cos\vartheta}{\sin^3\vartheta} \, \mathrm{d}\vartheta, \\
&= \frac{2\pi R^3}{3} \left(\tan\vartheta \Big|_0^{\tan^{-1}(h/R)} \right) + \frac{2\pi h^3}{3} \left(\frac{-1}{2\tan^2\vartheta} \Big|_{\tan^{-1}(h/R)}^{\pi/2} \right), \\
&= \frac{2\pi R^3}{3} \left(\frac{h}{R} \right) + \frac{2\pi h^3}{3} \left(0 + \frac{1}{2h^2/R^2} \right), \\
&= \frac{2\pi R^2 h}{3} + \frac{\pi R^2 h}{3}, \\
&= \pi R^2 h.
\end{aligned}$$

Sphere: The sphere is much easier, because it is a surface of uniform $\mathfrak{z}$. Let's choose $r_0 = R$, then

$$\begin{aligned}
V &= \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \int_{-R}^{0} (\mathfrak{z} + R)^2 \cos\vartheta \, \mathrm{d}\mathfrak{z} \, \mathrm{d}\phi \, \mathrm{d}\vartheta, \\
&= \frac{2\pi R^3}{3} \int_{-\pi/2}^{\pi/2} \cos\vartheta \, \mathrm{d}\vartheta, \\
&= \frac{4\pi R^3}{3}.
\end{aligned}$$

6 $dSphere/dr = Area$

Problem 5.4.25 Boas (2006). The volume inside a sphere of radius r is $V = \frac{4}{3}\pi r^3$. Then $\mathrm{dV} = 4\pi \mathrm{r}^2 \mathrm{dr} = \mathrm{Adr}$, where A is the area of the sphere. What is the geometrical meaning of the fact that the derivative of the volume is the area? Could you use this fact to find the volume formula given the area formula?

The volume of a sphere is $V = \frac{4}{3}\pi r^3$, so $\mathrm{d}V = 4\pi r^2 \,\mathrm{d}r = A\,\mathrm{d}r$. The volume of a sphere is related to the integral, with respect to r, of the surfaces of the spherical shells inside it. You can find the volume from the area by integrating $V = \int \mathrm{d}V = \int_0^r A\,\mathrm{d}r$.

7 Square Wave

Problem 7.5.1 of Boas (2006). Expand $f(x)$, which is 1 from $-\pi$ to 0 and 0 from 0 to π (and periodic after that), as a sine-cosine Fourier series.

We examine the averages of $f(x)$, which is 1 from $-\pi$ to 0 and 0 from 0 to π.

$$\begin{aligned}
\langle f(x)\cos 0x\rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)1\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{0} 1\,\mathrm{d}x = \frac{1}{2},\\
\langle f(x)\cos x\rangle &= \frac{1}{2\pi}\int_{-\pi}^{0} \cos x\,\mathrm{d}x = \frac{\sin(0)-\sin(-\pi)}{2\pi} = 0,\\
\langle f(x)\cos nx\rangle &= \frac{1}{2\pi}\int_{-\pi}^{0} \cos nx\,\mathrm{d}x = \frac{\sin(n\pi)}{2\pi n},\\
\langle f(x)\sin x\rangle &= \frac{1}{2\pi}\int_{-\pi}^{0} \sin x\,\mathrm{d}x = \frac{-\cos 0+\cos(-\pi)}{2\pi} = \frac{-1}{\pi},\\
\langle f(x)\sin nx\rangle &= \frac{1}{2\pi}\int_{-\pi}^{0} \sin nx\,\mathrm{d}x = \frac{-1+\cos(n\pi)}{2\pi n}.
\end{aligned}$$

So, we want to expand the function in a series, which we do by multiplying and then averaging both sides,

$$\begin{aligned}
f(x) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx,\\
\langle f(x)\rangle &= \frac{1}{2} = \langle\cos x\left(\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx\right)\rangle = \frac{1}{2}a_0 \to a_0 = 1,\\
\langle f(x)\cos x\rangle &= 0 = \langle\cos x\left(\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx\right)\rangle = \frac{1}{2}a_1 \to a_1 = 0,\\
\langle f(x)\cos nx\rangle &= \frac{\sin(n\pi)}{2\pi n} = \langle\cos nx\left(\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx\right)\rangle = \frac{1}{2}a_n \to a_n = \frac{\sin(n\pi)}{\pi n} = 0,\\
\langle f(x)\sin x\rangle &= \frac{-1}{\pi} = \langle\sin x\left(\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx\right)\rangle = \frac{1}{2}b_1 \to b_1 = \frac{-2}{\pi},\\
\langle f(x)\sin nx\rangle &= \frac{-1+\cos(n\pi)}{2\pi n} = \langle\sin nx\left(\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx\right)\rangle = \frac{1}{2}b_n \to b_n = \frac{-1+\cos(n\pi)}{\pi n},\\
f(x) &= \frac{1}{2} + \sum_{n=1}^{\infty}\frac{-1+\cos(n\pi)}{\pi n}\sin nx = \frac{1}{2} - \frac{2}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right]
\end{aligned}$$

8 Sines, Cosines, Exponentials

Problem 7.5.12 of Boas (2006). Show that in (5.2) the average values of $\sin(mx)\sin(nx)$ and of $\cos(mx)\cos(nx)$, $m \neq n$, are zero (over a period), by using the complex exponential forms for the sines and cosines as in (5.3).

$$\begin{aligned}
\langle \sin nx \sin mx \rangle &= -\frac{1}{4}\left\langle \left(e^{inx} - e^{-inx}\right)\left(e^{imx} - e^{-imx}\right)\right\rangle, \\
&= -\frac{1}{4}\left\langle e^{inx+imx} - e^{-inx+imx} - e^{inx-imx} + e^{-inx-imx}\right\rangle, \\
&= \begin{cases} -\frac{1}{4}(1-0-0+1) & \text{if } m = -n \neq 0 \\ -\frac{1}{4}(0-1-1+0) & \text{if } m = n \neq 0 \\ -\frac{1}{4}(0-0-0+0) & \text{if } m \neq \pm n \\ -\frac{1}{4}(1-1-1+1) & \text{if } m = n = 0 \end{cases} = \begin{cases} 0, & \text{if } m = n = 0, \\ 0, & \text{if } m \neq n, \\ \frac{1}{2}, & \text{if } m = n \neq 0. \end{cases}
\end{aligned}$$

The last equality takes advantage of the oddness of sine.

$$\begin{aligned}
\langle \sin nx \sin mx \rangle &= \frac{1}{4}\left\langle \left(e^{inx} + e^{-inx}\right)\left(e^{imx} + e^{-imx}\right)\right\rangle, \\
&= \frac{1}{4}\left\langle e^{inx+imx} + e^{-inx+imx} + e^{inx-imx} + e^{-inx-imx}\right\rangle, \\
&= \begin{cases} \frac{1}{4}(1+0+0+1) & \text{if } m = -n \neq 0 \\ \frac{1}{4}(0+1+1+0) & \text{if } m = n \neq 0 \\ \frac{1}{4}(0+0+0+0) & \text{if } m \neq \pm n \\ \frac{1}{4}(1+1+1+1) & \text{if } m = n = 0 \end{cases} = \begin{cases} 1, & \text{if } m = n = 0, \\ 0, & \text{if } m \neq n, \\ \frac{1}{2}, & \text{if } m = n \neq 0. \end{cases}
\end{aligned}$$

Quod erat demonstrum.

9 Derivatives

a) Show that the following function $f(x)$ and Fourier series $g(x)$ are equivalent on the interval from $-\pi$ to π up to order of $\sin(2x)$. To do so, multiply the $f(x)$ and $g(x)$ functions by each of the following in turn: $\sin(x), \sin(2x)$ and $\cos(0x), \cos(x), \cos(2x)$. Show that the average value of the product from $-\pi$ to π is the same, for example that $\langle f(x)\sin(2x)\rangle = \langle g(x)\sin(2x)\rangle$. (see Boas, 2006, pg. 351).

$$\begin{aligned}
\forall : -\pi \leq x \leq \pi&, \\
f(x) &= x(\pi - x)(\pi + x), \\
g(x) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} 12 \sin(nx).
\end{aligned}$$

b) Take the first derivative of $f(x)$ and $g(x)$ (by taking the derivative of the generic term in the series). Show that the resulting derivatives are equivalent, using the same method as in a).

(a) Exploiting the fact that any odd function will be zero on average, and explicitly calculating the even functions, we find

$$\langle f(x)\rangle = \langle x(\pi - x)(\pi + x)\rangle = 0, \qquad \langle g(x)\rangle = 0.$$
$$\langle f(x)\cos x\rangle = \langle x(\pi - x)(\pi + x)\cos x\rangle = 0, \qquad \langle g(x)\cos x\rangle = 0.$$
$$\langle f(x)\cos mx\rangle = \langle x(\pi - x)(\pi + x)\cos mx\rangle = 0, \qquad \langle g(x)\cos mx\rangle = 0.$$
$$\langle f(x)\sin x\rangle = \langle x(\pi - x)(\pi + x)\sin x\rangle \qquad \langle g(x)\sin x\rangle = 6$$
$$= \langle(\pi^2 x - x^3)\sin x\rangle = \pi^2 - (-6 + \pi^2) = 6$$
$$\langle f(x)\sin mx\rangle = \langle x(\pi - x)(\pi + x)\sin mx\rangle \qquad \langle g(x)\sin mx\rangle = \frac{(-1)^{m-1}}{m^3}6.$$
$$= \langle(\pi^2 x - x^3)\sin nx\rangle = \frac{-6\cos m\pi}{n^3} = \frac{-6(-1)^m}{m^3}$$

(b) Now we check the derivative...

$$f'(x) = (\pi^2 - 3x^2),$$
$$g'(x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}12\cos(nx).$$
$$\langle f'(x)\rangle = \langle(\pi^2 - 3x^2)\rangle = \pi^2 - \pi^2 = 0, \qquad \langle g'(x)\rangle = 0.$$
$$\langle f'(x)\cos x\rangle = \langle(\pi^2 - 3x^2)\cos x\rangle = 0 - (-6), \qquad \langle g'(x)\cos x\rangle = 6.$$
$$\langle f'(x)\cos mx\rangle = \langle(\pi^2 - 3x^2)\cos mx\rangle \qquad \langle g'(x)\cos mx\rangle = \frac{-6(-1)^m}{m^2}.$$
$$= 0 - \frac{6\cos m\pi}{m^2} = \frac{-6(-1)^m}{m^2}$$
$$\langle f'(x)\sin x\rangle = 0 \qquad \langle g'(x)\sin x\rangle = 0$$
$$\langle f'(x)\sin mx\rangle = 0 \qquad \langle g'(x)\sin mx\rangle = 0.$$

Quod erat demonstrum.

References

Boas, M. L., 2006. Mathematical methods in the physical sciences, 3rd Edition. Wiley, Hoboken, NJ.