Fall 2019 GEOL0350–Homework 5
Based on Chapter 8 of the Notes (Vectors)

1 Centripetal

Problem 6.3.17 Boas (2006). 6.3.17: Expand the triple product $\mathbf{a} = \omega \times \omega \times \mathbf{r}$ given in the discussion of this figure. If $\mathbf{r}$ is perpendicular to ω (Problem 16), show that $a = -\omega^2\mathbf{r}$, and so find the elementary result that the acceleration is toward the center of the circle and of magnitude v^2/r.

Applications of the Triple Vector Product In Figure 3.8 (compare Figure 2.6), suppose the particle m is at rest on a rotating rigid body (for example, the earth). Then the *angular momentum* $\mathbf{L}$ of m about point O is defined by the equation $\mathbf{L} = \mathbf{r} \times (m\mathbf{v}) = m\mathbf{r} \times \mathbf{v}$. In the discussion of Figure 2.6, we showed that $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$. Thus, $\mathbf{L} = m\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})$. See Problem 16 and also Chapter 10, Section 4.

As another example, it is shown in mechanics that the centripetal acceleration of m in Figure 3.8 is $\mathbf{a} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$. See Problem 17.

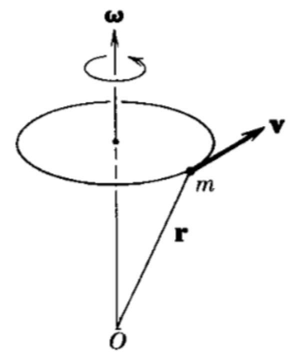

Figure 3.8

From (**??**), and if $\mathbf{r}$ is perpendicular to ω, we have

$$\begin{aligned}\mathbf{a} &= \omega \times (\omega \times \mathbf{r}) = \omega\,(\omega \cdot \mathbf{r}) - \mathbf{r}\,(\omega \cdot \omega)\,,\\ &= -\mathbf{r}\,(\omega \cdot \omega) = -\mathbf{r}\omega^2 = -\hat{\mathbf{r}}\frac{\omega^2 r^2}{r} = -\hat{\mathbf{r}}\frac{v_\phi^2}{r}.\end{aligned}$$

2 Direction of Decrease

Problem 6.6.2 of Boas (2006). Starting from the point (1,1), in what direction does the function $\phi = x^2 - y^2 + 2xy$ *decrease* most rapidly?

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$$\nabla\phi = (2x+2y)\hat{\mathbf{i}} + (-2y+2x)\hat{\mathbf{j}},$$
$$\nabla\phi\bigg|_{(x,y)=(1,1)} = 4\hat{\mathbf{i}}.$$

Thus, the direction of greatest increase in ϕ is in the $\hat{\mathbf{i}}$ direction. The direction of greatest decrease is therefore in the opposite direction, or the $-\hat{\mathbf{i}}$ direction.

3 Calculate Div, Grad, Curl

Problem 6.7.1 Boas (2006). Compute the divergence and curl of $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$.

Compute the divergence and curl of $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$.

$$\nabla\cdot\mathbf{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3,$$
$$\nabla\times\mathbf{r} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \hat{\mathbf{i}}(0) + \hat{\mathbf{j}}(0) + \hat{\mathbf{k}}(0) = 0.$$

4 A Triple Product

Problem 6.7.18 Boas (2006). For $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, evaluate $\nabla\times(\hat{\mathbf{k}}\times\mathbf{r})$

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}},$$
$$\nabla\times(\hat{\mathbf{k}}\times\mathbf{r}) = \nabla\times\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 1 \\ x & y & z \end{vmatrix} = \nabla\times\left(-y\hat{\mathbf{i}} + x\hat{\mathbf{j}}\right) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = 2\hat{\mathbf{k}}.$$

Alternatively, you could use the product rule,

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}),$$
$$\nabla\times\left(\hat{\mathbf{k}}\times\mathbf{r}\right) = \cancel{(\mathbf{r}\cdot\nabla)\hat{\mathbf{k}}}^{0} - (\hat{\mathbf{k}}\cdot\nabla)\mathbf{r} + \hat{\mathbf{k}}(\nabla\cdot\mathbf{r}) - \cancel{\mathbf{r}(\nabla\cdot\hat{\mathbf{k}})}^{0},$$
$$= -\frac{\partial\mathbf{r}}{\partial z} + 3\hat{\mathbf{k}} = -\hat{\mathbf{k}} + 3\hat{\mathbf{k}} = 2\hat{\mathbf{k}}.$$

5 Check for Conservative–Attraction to the origin

Problem 6.8.10 of Boas (2006). This problem was selected because it is very similar to Hooke's law, which is a fundamental of solid mechanics and gravitational attraction. Verify that the following force field is conservative. Then find, for each, a scalar potential ϕ such that $F = -\nabla\phi$.

$$\mathbf{F} = -k\mathbf{r}, \tag{1}$$
$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}, \tag{2}$$
$$k = \text{constant}. \tag{3}$$

$$\mathbf{F} = -k\mathbf{r}, \qquad \mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}, \qquad k = \text{constant},$$
$$\nabla\times\mathbf{F} = -k\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \hat{\mathbf{i}}(0) + \hat{\mathbf{j}}(0) + \hat{\mathbf{k}}(0) = 0, \text{ or,}$$
$$\nabla\times\mathbf{F} = -k\left[\nabla\times\mathbf{r}\right] = -k\left[\frac{\hat{\mathbf{r}}}{r\sin\theta}\left[\frac{\partial(\sin\theta v_\phi)}{\partial\theta} - \frac{\partial v_\theta}{\partial\phi}\right] + \frac{\hat{\theta}}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial(r v_\phi)}{\partial r}\right] + \frac{\hat{\phi}}{r}\left[\frac{\partial(r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial\theta}\right]\right] = 0.$$

So, $\mathbf{F}$ is conservative, and therefore can be generated as the gradient of a scalar. Now, to find a scalar whose gradient is $\mathbf{F}$. We notice that lines of equal $\mathbf{F}$ are concentric rings. We can therefore the gradient rule, to effectively generate the function by a line integral from the origin outward, which is just a regular integration in dr.

$$\int_0^r -\mathbf{F}\cdot \mathrm{d}\mathbf{l} = \int_0^r \nabla\phi\,\mathrm{d}\mathbf{l} = \phi(r) - \phi(0),$$
$$\int_0^r -\mathbf{F}\cdot \mathrm{d}\mathbf{l} = \int_0^r kr\,\mathrm{d}r = \frac{1}{2}kr^2, \text{ so,,}$$
$$\phi(r) = \frac{1}{2}kr^2.$$

Where in the last step we notice that any constant we choose for $\phi(0)$ will not affect the fact that $\mathbf{F} = -\nabla\phi$, so we can choose $\phi(0) = 0$.

6 Gradient of r

The combination of gravitation and the centrifugal force from the earth's rotation is a conservative force that can be expressed using the geopotential $\phi = mgz$, where z is distance from the surface and m and g are the constant mass and acceleration due to gravity. A motion that results in a change in geopotential indicates the possibility that energy can be extracted from the motion. Formulate a closed line integral (using Stokes theorem) for a route to school that proves that "in my day, we had to go to school uphill both ways!" cannot require a net expenditure of energy. Show that if there is a nonconservative force (e.g., viscosity of air, rusty bike wheels, etc.) there may be a nonzero expenditure of energy in the round trip.

There are two ways to solve this problem. The first uses the gradient theorem. We define "uphill" as walking in a direction opposing a force (i.e., work is being done by you). Therefore, the amount of work (average walking uphill) from point $\mathbf{a}$ to point $\mathbf{b}$ is

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot \mathrm{d}\mathbf{l} = \int_{\mathbf{a}}^{\mathbf{b}} -\nabla\phi \cdot \mathrm{d}\mathbf{l} = \phi(\mathbf{a}) - \phi(\mathbf{b}).$$

And if you are going in a roundtrip to school and back, then $\mathbf{a} = \mathbf{b}$, so no work is done and you cannot have gone uphill both ways. To put it another way,

$$0 = \phi(\mathbf{a}) - \phi(\mathbf{b}) - \phi(\mathbf{b}) + \phi(\mathbf{a}) = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot \mathrm{d}\mathbf{l} + \int_{\mathbf{b}}^{\mathbf{a}} \mathbf{F} \cdot \mathrm{d}\mathbf{l}.$$

So, if either of the last two is negative (i.e., uphill on average), then the other must be positive (downhill on average).
The second way to do this is to consider the curl theorem over any closed path from school to home and back.

$$\oint \mathbf{F} \cdot \mathrm{d}\mathbf{l} = \iint \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}A = -\iint \nabla \times \nabla\phi \cdot \hat{\mathbf{n}}\, \mathrm{d}A = 0.$$

The last is true because the curl of any gradient is zero. The first integral can be interpreted as the average amount of uphill over the whole circuit from home to school.
Notice that *neither of these solutions depends on the path chosen*, which is nice for debunking the storyteller (well, on the path we took to school…).
If there is a nonconservative force, then we can modify the Stokes theorem version trivially,

$$\oint \mathbf{F} \cdot \mathrm{d}\mathbf{l} = \iint \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}A = -\iint \nabla \times (\nabla\phi + \mathbf{F}_{nc}) \cdot \hat{\mathbf{n}}\, \mathrm{d}A = -\iint \nabla \times \mathbf{F}_{nc} \cdot \hat{\mathbf{n}}\, \mathrm{d}A.$$

The nonconservative force might (or might not, depending on the path) contribute a nonzero amount of work required. Typical nonconservative forces are like walking through molasses, in which case the force always opposes the motion.

7 Stokes Theorem

a) Calculate the curl of the vector $\mathbf{v} = (-y, x, 0)$. b) Take the area integral of $(\nabla \times \mathbf{v}) \cdot \hat{\mathbf{n}}$ over the surface of a disc bounded by $1 = x^2 + y^2$. c) Take the area integral of $(\nabla \times \mathbf{v}) \cdot \hat{\mathbf{n}}$ over the surface of the half-sphere bounded by $1 = x^2 + y^2 + z^2$ where $z \geq 0$. d) Use Stokes' theorem to find a line integral equal to both b) and c).

$$(a)\nabla\times\mathbf{v}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ -y & x & 0\end{vmatrix}=2\hat{\mathbf{k}}.,$$

$$(b)Cylindrical:\int_0^{2\pi}\int_0^1 2\hat{\mathbf{k}}\cdot\hat{\mathbf{n}}r\,\mathrm{d}r\,\mathrm{d}\phi=\int_0^{2\pi}\int_0^1 2r\,\mathrm{d}r\,\mathrm{d}\phi=2\pi,$$

$$(c)Spherical:\int_0^{2\pi}\int_0^{\pi/2} 2\hat{\mathbf{k}}\cdot\hat{\mathbf{n}}\,\cancel{r^2}^{1}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi=\int_0^{2\pi}\int_0^{\pi/2} 2\cos\theta\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi=\int_0^{2\pi}\mathrm{d}\phi=2\pi,$$

$$(d)Either:\oint\mathbf{v}\cdot\mathrm{d}\mathbf{l}=\int_0^{2\pi}\mathbf{v}\cdot\hat{\phi}r\,\mathrm{d}\phi=\int_0^{2\pi}(-r\sin\phi[\hat{\mathbf{r}}\cos\phi-\hat{\phi}\sin\phi]+r\cos\phi[\hat{\mathbf{r}}\sin\phi+\hat{\phi}\cos\phi])\cdot\hat{\phi}r\,\mathrm{d}\phi,$$

$$\text{as }\hat{\mathbf{i}}=\hat{\mathbf{r}}\cos\phi-\hat{\phi}\sin\phi,\qquad \hat{\mathbf{j}}=\hat{\mathbf{r}}\sin\phi+\hat{\phi}\cos\phi\qquad x=r\cos\phi,\qquad y=r\sin\phi,$$

$$\oint\mathbf{v}\cdot\mathrm{d}\mathbf{l}=\int_0^{2\pi}(r\sin^2\phi+r\cos^2\phi)r\,\mathrm{d}\phi=2\pi.$$

8 Divergence Theorem

a) Calculate the divergence of the vector $\mathbf{v}=(-y,x,1)$. b) What is the area integral of $\oiint\mathbf{v}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$, over any given closed area? c) Can you say anything about its integral over an area that is not closed?

$$(a)\nabla\cdot\mathbf{v}=0.$$
$$(b)\oiint\mathbf{v}\cdot\hat{\mathbf{n}}\,\mathrm{d}S=\iiint\cancel{\nabla\cdot\mathbf{v}}^{0}\,\mathrm{d}V=0.$$

(c) Not much. If the surface integral is open, there can be a large flow out through it, which can allow a large inflow elsewhere through the part of the surface you are integrating over. Think of puncturing a water balloon with a needle–a lot of flow can leave through a tiny hole even though the water is nearly incompressible and most of the surface integral (everywhere not punctured) is zero.

References

Boas, M. L., 2006. Mathematical methods in the physical sciences, 3rd Edition. Wiley, Hoboken, NJ.