# Fall 2019 GEOL0350–Homework 6
# Based on Chapter 9 of the Notes (ODEs)

## 1 Separate & Conquer

Problem 8.2.1 of Boas (2006). For the following differential equation, separate variables (collect all $x, \mathrm{d}x$ on one side and all $y, \mathrm{d}y$ on the other) and integrate to find a solution containing one arbitrary constant. Then find the value of the constant to give a particular solution satisfying the given boundary condition ($y = 3$ when $x = 2$).

$$xy' = y \tag{1}$$

$$\begin{aligned} x\frac{\mathrm{d}y}{\mathrm{d}x} &= y, \\ \frac{\mathrm{d}y}{y} &= \frac{\mathrm{d}x}{x}, \\ \int \frac{\mathrm{d}y}{y} &= \int \frac{\mathrm{d}x}{x}, \\ \ln y = \ln x + \ln C&, \\ \ln y = \ln Cx&, \\ y = Cx&. \end{aligned}$$

Where $C$ is an arbitrary constant. The second step is to evaluate $C$ such that $y = 3$ when $x = 2$, thus

$$y = \frac{3}{2}x$$

## 2 Guess & Check

Problem 8.5.1 of Boas (2006). Solve the following differential equation by guess-and-check method, assuming an exponential form: $y = Ae^{mx}$. This form is useful when the equation is *homogeneous* with constant coefficients.

$$y'' + y' - 2y = 0 \tag{2}$$

$$\begin{aligned} \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 0, \\ y = Ae^{mx}, \\ m^2 + m - 2 = 0 \to m = \frac{-1 \pm \sqrt{1+8}}{2} = -2 \text{ or} 1, \\ y = Ae^{-2x} + Be^{1x}. \end{aligned}$$

# 3 Evaluating the General

Show that the general solution to the first-order, linear differential equation in (9.2) of the notes:

$$y = e^{-\int P(x)\,\mathrm{d}x} \int \left( Q(x) e^{\int P(x)\,\mathrm{d}x}\,\mathrm{d}x \right) + Ce^{-\int P(x)\,\mathrm{d}x}. \tag{3}$$

is the same as the guess & check solution in (9.24) of the notes:

$$\langle \mathcal{P} \rangle \approx \langle \mathcal{P}_{ss} \rangle + Ae^{-t \frac{V_{out}}{\text{Vol.}}} = \langle \mathcal{P}_{ss} \rangle \left( 1 - e^{-t \frac{V_{out}}{\text{Vol.}}} \right)$$

For the equation

$$\frac{d}{dt} \langle \mathcal{P} - \mathcal{P}_{ss} \rangle + \frac{V_{out}}{\text{Vol.}} \langle \mathcal{P} - \mathcal{P}_{ss} \rangle \approx 0$$

The solution in (9.24) is

$$\langle \mathcal{P} \rangle \approx \langle \mathcal{P}_{ss} \rangle + Ae^{-t \frac{V_{out}}{\text{Vol.}}} = \langle \mathcal{P}_{ss} \rangle \left( 1 - e^{-t \frac{V_{out}}{\text{Vol.}}} \right)$$

So, let's work on the general solution. The problem is stated as

$$y' + Py = Q.$$

The solution is formally,

$$\begin{aligned} ye^I &= \int Qe^I\,\mathrm{d}x + c, \qquad \text{or} \\ y &= e^{-I} \int Qe^I\,\mathrm{d}x + ce^{-I}, \qquad \text{where} \\ I &= \int P\,\mathrm{d}x. \end{aligned}$$

Our problem is

$$\frac{d}{dt} \langle \mathcal{P} - \mathcal{P}_{ss} \rangle + \frac{V_{out}}{\text{Vol.}} \langle \mathcal{P} - \mathcal{P}_{ss} \rangle \approx 0$$

Thus,

$$\begin{aligned} y &= \langle \mathcal{P} - \mathcal{P}_{ss} \rangle, \\ x &= t, \\ P &= \frac{V_{out}}{\text{Vol.}}, \\ Q &= 0, \\ I &= \int \frac{V_{out}}{\text{Vol.}}\, \mathrm{d}t = \frac{V_{out} t}{\text{Vol.}} \end{aligned}$$

Thus, the solution is

$$y = \langle \mathcal{P} - \mathcal{P}_{ss} \rangle = e^{-I} \int Q e^{I}\, \mathrm{d}x + c e^{-I} = c e^{-\frac{V_{out}}{\text{Vol.}} t} = A e^{-t \frac{V_{out}}{\text{Vol.}}}$$

Which is the same as the solution given in (9.24).

# 4 Third Solution

Find the leading order differential equation, by Taylor series expansion of the radiation forcing, to the Energy Balance Model of Section 9.2.3 for the remaining steady state solution (near $T_{ss3} = 174.438\,\text{K}$). Solve this differential equation, and decide if this third steady state is a stable or unstable steady state solution.

Evaluating the Taylor series near $T_{ss3}$,

$$\begin{aligned} R_i &= 51.375\,\text{W}\,\text{m}^{-2}\,(1 + \dots)\,, \\ -R_o &= 51.375\,\text{W}\,\text{m}^{-2}\,\big(-1 - 0.02218(T - T_{ss3})\big)\,, \\ \frac{\mathrm{d}T}{\mathrm{d}t} &\approx \underbrace{\frac{-1.1395\,\text{W}\,\text{m}^{-2}\,\text{K}^{-1}}{c}}_{\text{neg. feedback}}(T - T_{ss3}) + \dots \\ T - T_{ss3} &\approx \Delta T e^{-t/\tau}, \qquad \tau = \frac{1.1395\,\text{W}\,\text{m}^{-2}\,\text{K}^{-1}}{c} \end{aligned}$$

This steady-state solution is stable.

# 5 Characteristic plus Particular

Verify that the sum of the characteristic solution (9.49) and the particular solution (9.44) constitute a solution to the original equations for the two sliding masses (9.43).

$$
\begin{aligned}
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} x_{p1} \\ x_{p2} \end{bmatrix} + \begin{bmatrix} x_{c1} \\ x_{c2} \end{bmatrix}, \\
\begin{bmatrix} x_{p1} \\ x_{p2} \end{bmatrix} &= \begin{bmatrix} Vt - D - 2\mu S/k \\ Vt - 2D - 3\mu S/k \end{bmatrix}, \\
\begin{bmatrix} x_{c1} \\ x_{c2} \end{bmatrix} &= (Ae^{i\omega_1 t} + Be^{-i\omega_1 t}) \begin{bmatrix} 1+\sqrt{5} \\ -2 \end{bmatrix} + \left(Ce^{i\omega_2 t} + De^{-i\omega_2 t}\right) \begin{bmatrix} 1-\sqrt{5} \\ -2 \end{bmatrix}, \\
\omega_1 &= \sqrt{\frac{k}{m}}\sqrt{\frac{3+\sqrt{5}}{2}}, \qquad \omega_2 = \sqrt{\frac{k}{m}}\sqrt{\frac{3-\sqrt{5}}{2}}
\end{aligned}
$$

So,

$$
\begin{aligned}
x_1 &= Vt - D - 2\mu S/k + \left[1+\sqrt{5}\right](Ae^{i\omega_1 t} + Be^{-i\omega_1 t}) + \left[1-\sqrt{5}\right]\left(Ce^{i\omega_2 t} + De^{-i\omega_2 t}\right), \\
x_2 &= Vt - 2D - 3\mu S/k - 2(Ae^{i\omega_1 t} + Be^{-i\omega_1 t}) - 2\left(Ce^{i\omega_2 t} + De^{-i\omega_2 t}\right).
\end{aligned}
$$

Our overall governing equations are:

$$
\begin{aligned}
m\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} &= \left(k\left[Vt - x_1 - D\right] - k\left[x_1 - x_2 - D\right] - \mu S\right), \\
m\frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} &= \left(k\left[x_1 - x_2 - D\right] - \mu S\right).
\end{aligned}
$$

From our solutions, we have

$$
\begin{aligned}
\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} &= -\omega_1^2\left[1+\sqrt{5}\right](Ae^{i\omega_1 t} + Be^{-i\omega_1 t}) - \omega_2^2\left[1-\sqrt{5}\right]\left(Ce^{i\omega_2 t} + De^{-i\omega_2 t}\right), \\
\frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} &= 2\omega_1^2(Ae^{i\omega_1 t} + Be^{-i\omega_1 t}) + 2\omega_2^2\left(Ce^{i\omega_2 t} + De^{-i\omega_2 t}\right), \\
k[Vt - x_1 - D] &= 2\mu S - k\left[1+\sqrt{5}\right](Ae^{i\omega_1 t} + Be^{-i\omega_1 t}) - k\left[1-\sqrt{5}\right]\left(Ce^{i\omega_2 t} + De^{-i\omega_2 t}\right), \\
k\left[x_1 - x_2 - D\right] &= \mu S + k\left[3+\sqrt{5}\right](Ae^{i\omega_1 t} + Be^{-i\omega_1 t}) + k\left[3-\sqrt{5}\right]\left(Ce^{i\omega_2 t} + De^{-i\omega_2 t}\right).
\end{aligned}
$$

$$
\begin{aligned}
\left(k\left[Vt - x_1 - D\right] - k\left[x_1 - x_2 - D\right] - \mu S\right) &= -k\left[4+2\sqrt{5}\right](Ae^{i\omega_1 t} + Be^{-i\omega_1 t}) - k\left[4-2\sqrt{5}\right]\left(Ce^{i\omega_2 t} + De^{-i\omega_2 t}\right), \\
\left(k\left[x_1 - x_2 - D\right] - \mu S\right) &= k\left[3+\sqrt{5}\right](Ae^{i\omega_1 t} + Be^{-i\omega_1 t}) + k\left[3-\sqrt{5}\right]\left(Ce^{i\omega_2 t} + De^{-i\omega_2 t}\right), \\
\frac{3+\sqrt{5}}{2}(1+\sqrt{5}) &= 4 + 2\sqrt{5}, \\
\frac{3-\sqrt{5}}{2}(1-\sqrt{5}) &= 4 - 2\sqrt{5}.
\end{aligned}
$$

These results, when assembled, prove the governing equations are satisfied by the solutions found.

# References

Boas, M. L., 2006. Mathematical methods in the physical sciences, 3rd Edition. Wiley, Hoboken, NJ.