

Fall 2019 GEOL0350–Homework 7

Based on Chapter 10 of the Notes (Dynamics & Chaos)

1 Fix & Perturb

Locate the fixed points of the following one-dimensional functions and use linear stability analysis and phase planes (\dot{x} vs. x) to categorize them. Consider only real values of a and x . a) $\dot{x} = a - e^{-x^2}$, b) $\dot{x} = ax - x^5$ (note that a can be of positive, zero, or negative—consider all 3). a) See Fig. ??.

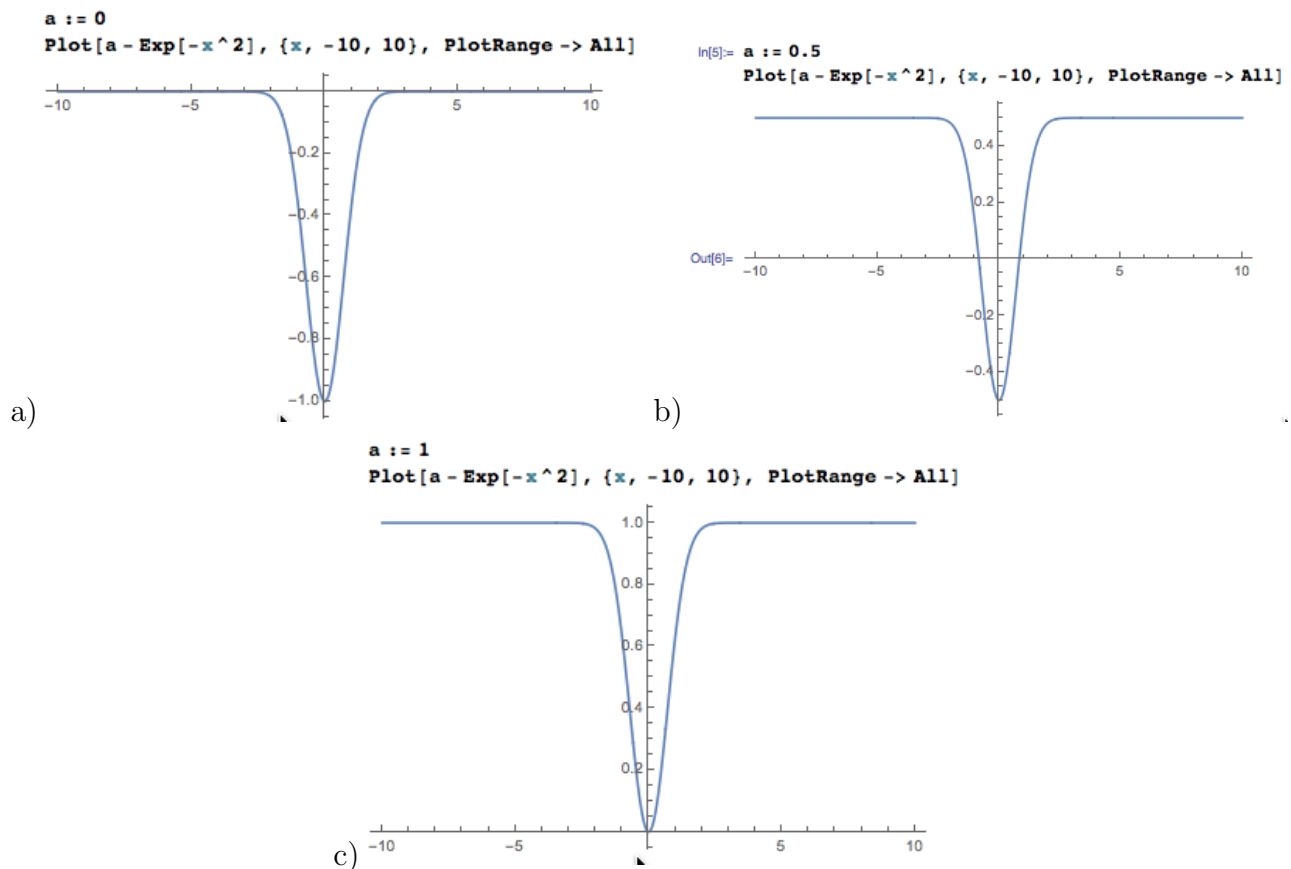


Figure 1: Plots of $\dot{x} = a - e^{-x^2}$ versus x varying a . In a), $a = 0$, where there are two fixed points at $\pm\infty$, with the one at $-\infty$ being stable and the one at ∞ being unstable. b) Shows when $a = 0.5$, where the fixed points are at $x = \pm\sqrt{-\ln a} = \pm 0.83255\dots$. The one at $x < 0$ is stable and the other is unstable. c) Shows the one half-stable fixed point when $a = 1$ at $x = 0$. There are no real fixed points when $a > 1$ or $a < 0$.

See Fig. ??

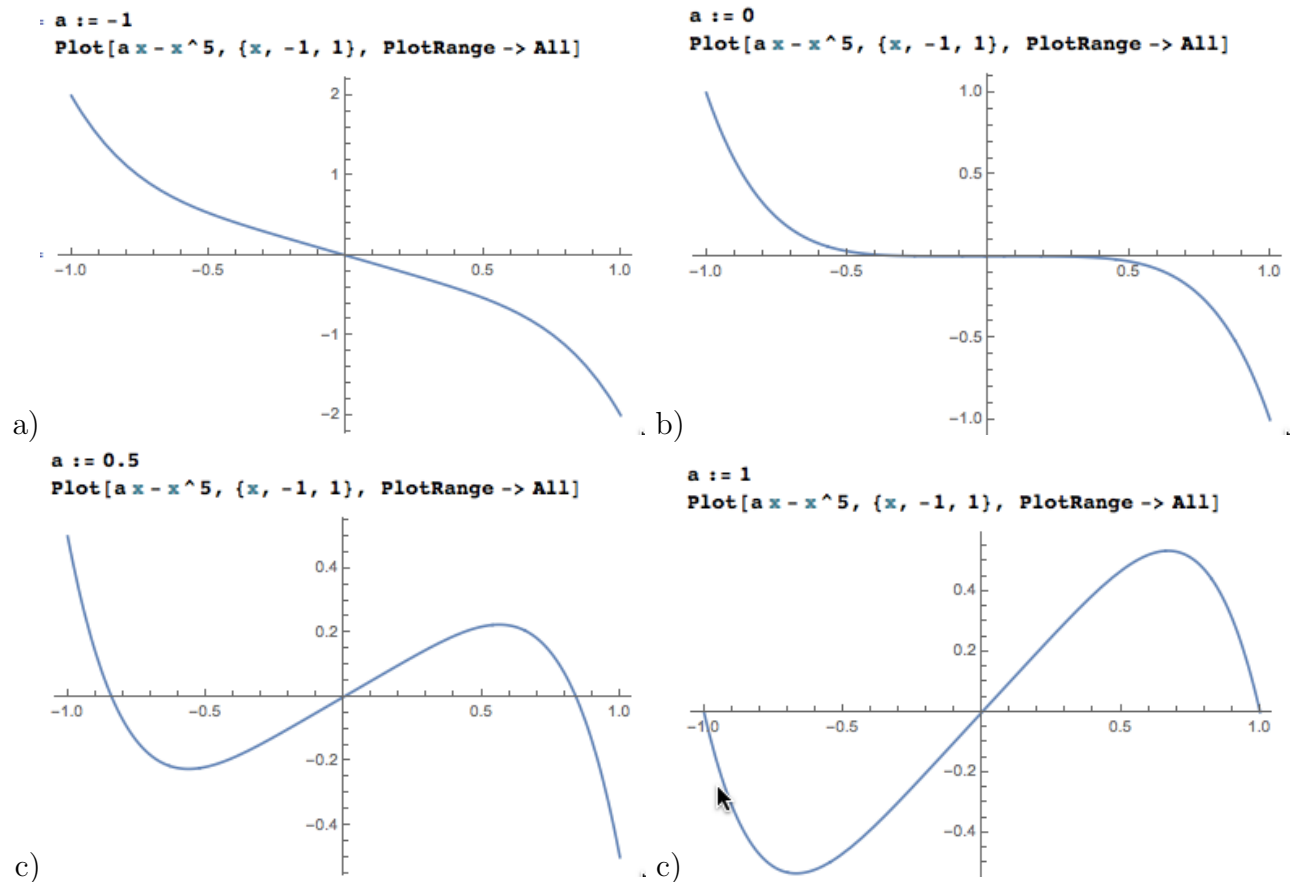


Figure 2: Plots of $\dot{x} = ax - x^5 = x(a - x^4)$ versus x varying a . In a), $a = -1$, where one stable root is at $x = 0$, with the rest being imaginary. b) Shows when $a = 0$, where all five roots are at $x = 0$, again this point is stable (although it is neutral—i.e., no slope—for the first 4 derivatives). c) Now there are three fixed points at $a = 0.5$. The one at $x = 0$ is unstable, and the other two are stable (pitchfork bifurcation). The two nonzero real roots are at $x = \pm \sqrt[4]{a}$. d) The solutions at $a = 1$ is the qualitatively the same as $a = 0.5$.

2 Bifurcate

Draw the bifurcation diagrams (x^* vs. a) for the following functions as a varies: a) $\dot{x} = a - e^{-x^2}$, b) $\dot{x} = ax - x^5$.

a) See Fig. ?? . b) See Fig. ??

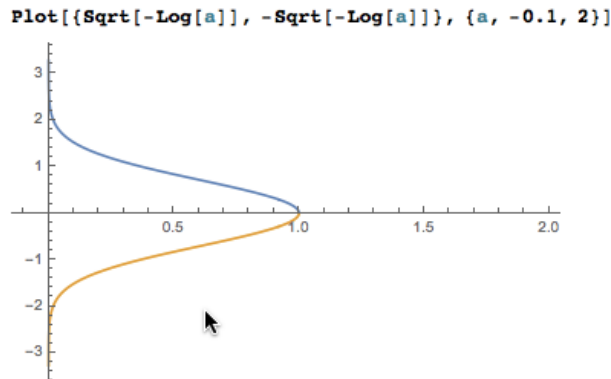


Figure 3: Bifurcation diagram of $0 = a - e^{-x^2}$ versus a . Upper branch is unstable, lower is stable.

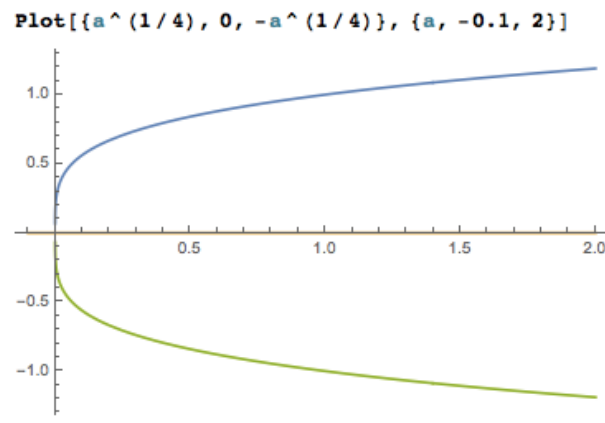


Figure 4: Bifurcation diagram of $0 = ax^* - x^{*5} = x^*(a - x^{*4})$ versus a . Nonzero branches are stable, branch at zero is stable for $a < 0$ and unstable for $a > 0$.

2.1 Stommel is Hysterical

The [Stommel \(1961\)](#) model exhibits hysteresis (Section 10.4.6) when the forcing is varied. The [Zaliapin and Ghil \(2010\)](#) model (Section 9.2.3) also exhibits hysteresis when the incoming solar radiation is varied. Which of the the following systems has hysteresis as the parameter a is varied between -1 and 1 ? Why or why not?

$$a) \quad \dot{x} = ax - x^3, \quad (1)$$

$$b) \quad \dot{x} = a + 2x - x^3, \quad (2)$$

$$c) \quad \dot{x} = a - x^2 \quad (3)$$

Actually, none of these exhibit hysteresis for a ranging between -1 and 1 . What are the requirements for hysteresis? Multiple solutions in x for a given a (so that one can be close to one or the other as a is varying.), and a disappearance or reappearance of solutions with varying a that tends to make the solution jump from one branch to another in a different direction depending on whether a is increasing or decreasing. Let's take each in turn.

a) This is our prototypical pitchfork bifurcation. It has one stable root for $a < 0$ and two stable and one unstable for $a > 0$. So, the possibility exists that we might come in from $a > 0$ on the upper branch, transition onto the $x = 0$ branch for $a < 0$, and then leave on the lower branch. However, this supposes that we somehow can cross over the $x = 0$ stable or unstable fixed point. That is, if we arrive at $a = 0$ just above $x = 0$, there is no way to get to negative x and return on the bottom branch. So, no hysteresis.

b) This system is the closest to hysteresis among those presented. The parametric plot has the classic S shape that leads to hysteresis. However, the bluesky bifurcations that make the upper two branches or the lower two branches disappear occurs slightly outside of this value ($a = \sqrt{32/27} \approx 1.08866$). To locate this point, note that the extrema of the function in x occur at $x = \pm\sqrt{2/3}$ (set derivative $\frac{\partial \dot{x}}{\partial x} = 0$ and solve). The value where $a + 2x - x^3 = 0$ when $x = \pm\sqrt{2/3}$ occurs when $a = \pm\sqrt{32/27}$. However, if a is varied between, e.g., -1.1 and 1.1 , this system will exhibit hysteresis.

c) This system has only two fixed points, and one is always unstable. Thus, there can be no hysteresis, as there are not two branches to jump between.

2.2 Phractals are Phun

Go to <http://usefuljs.net/fractals/> and <http://kevs3d.co.uk/dev/1systems> and make some pretty Julia, Mandelbrot, and fractal sets! Print out your favorites. Advanced: Try out the design a custom fractal! [You're on your own here!](#)

References

- Stommel, H. M., 1961. Thermohaline convection with 2 stable regimes of flow. *Tellus* 13, 224–230.
- Zaliapin, I., Ghil, M., 2010. Another look at climate sensitivity. *Nonlinear Processes In Geophysics* 17, 113–122.