Fall 2019 GEOL0350–Homework 8
Based on Chapter 11 of the Notes (PDEs)

1 Guess Wave and Check

Problem 13.1.2 of Boas (2006). (a) Show that the expression $u = \sin(x \pm vt)$ describing a sinusoidal wave, satisfies the wave equation $\frac{\partial^2 u}{\partial^2 t} - v^2 \frac{\partial^2 u}{\partial^2 x} = 0$. Show that, in general, $u = f(x - vt)$ and $u = f(x + vt)$ satisfy the wave equation, where f is any function with a second derivative. This is the d'Alembert solution of the wave equation. The function $f(x - vt)$ represents a wave moving in the positive x direction and $f(x + vt)$ represents a wave moving in the opposite direction.

(b) Show that $u(r, t) = (1/r)f(r - vt)$ and $u(r, t) = (1/r)f(r + vt)$ satisfy the wave equation in spherical coordinates $\frac{\partial^2 u}{\partial^2 t} - v^2\nabla^2 u = 0$. [Use the first term of the spherical coordinate Laplacian (8.39) for $\nabla^2 u$ since here u is independent of θ and ϕ.] These functions represent spherical waves spreading out from the origin or converging on the origin.

a) First check the D'Alembert solutions,

$$
\begin{aligned}
u = \sin(x - vt) \to \frac{\partial^2 u}{\partial x^2} &= -\sin(x - vt), & \frac{\partial^2 u}{\partial t^2} &= -v^2 \sin(x - vt), & \therefore \nabla^2 u &= \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}. \\
u = f(x - vt) \to \frac{\partial^2 u}{\partial x^2} &= f''(x - vt), & \frac{\partial^2 u}{\partial t^2} &= v^2 f''(x - vt), & \therefore \nabla^2 u &= \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}. \\
u = f(x + vt) \to \frac{\partial^2 u}{\partial x^2} &= f''(x + vt), & \frac{\partial^2 u}{\partial t^2} &= v^2 f''(x + vt), & \therefore \nabla^2 u &= \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}.
\end{aligned}
$$

b) Now, check on the spherical form.

$$
\begin{aligned}
u &= \frac{1}{r} f(r - vt) \to \nabla^2 u = ? \\
\nabla^2 u &= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial u}{\partial r}\right] = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(\frac{1}{r}f'(r - vt) - \frac{1}{r^2}f(r - vt)\right)\right] \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left[rf'(r - vt) - f(r - vt)\right] = \frac{1}{r}f''(r - vt), \\
\frac{\partial^2 u}{\partial t^2} &= \frac{v^2}{r}f(r - vt), \qquad \therefore \nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}. \\
u &= \frac{1}{r} f(r + vt) \to \nabla^2 u = ? \\
\nabla^2 u &= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial u}{\partial r}\right] = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(\frac{1}{r}f'(r + vt) - \frac{1}{r^2}f(r + vt)\right)\right] \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left[rf'(r + vt) - f(r + vt)\right] = \frac{1}{r}f''(r + vt), \\
\frac{\partial^2 u}{\partial t^2} &= \frac{v^2}{r}f(r + vt), \qquad \therefore \nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}.
\end{aligned}
$$

2 Steady Temperature

Problem 13.2.10 of Boas (2006). You do not need to make a computer plot of your results.

Find the steady-state temperature distribution in a metal plate 10 cm square if one side is held at 100° and the other three sides at 0°. Find the temperature at the center of the plate. (The answer, but not the solution method, is written out in Boas).

In class, we solved a boundary value problem for Laplace's equation with the boundary conditions: $\phi(L/2, y) = 0$, $\phi(-L/2, y) = 0$, $\phi(x, L) = 0$, $\phi(x, 0) = A\cos(\pi x/L)$. We can adapt our solution to examine $T(L, y) = 0$, $T(0, y) = 0$, $T(x, L) = 0$, $T(x, 0) = 100$.

$$\sum_k \left[A_{k+}e^{ikL} + A_{k-}e^{-ikL}\right]\left[B_{k+}e^{ky} + B_{k-}e^{-ky}\right] = 0, \tag{1}$$

$$\sum_k \left[A_{k+} + A_{k-}\right]\left[B_{k+}e^{ky} + B_{k-}e^{-ky}\right] = 0, \tag{2}$$

$$\sum_k \left[A_{k+}e^{ikx} + A_{k-}e^{-ikx}\right]\left[B_{k+}e^{kL} + B_{k-}e^{-kL}\right] = 0, \tag{3}$$

$$\sum_k \left[A_{k+}e^{ikx} + A_{k-}e^{-ikx}\right]\left[B_{k+} + B_{k-}\right] = 100. \tag{4}$$

Considering (**??**-**??**), these are satisfied if $A_{k+} = -A_{k-} = A_k/(2i)$, producing a sine series, and $\sin(kL)$ to be zero, which means that $k = \frac{\pi}{L}, \frac{3\pi}{L}, \cdots = n\pi/L$ where n is odd.

$$A_k \sin(kx)\left[B_{k+}e^{kL} + B_{k-}e^{-kL}\right] = 0 \leftrightarrow B_{k+}e^{kL} + B_{k-}e^{-kL} = 0$$

$$\sum_k A_k \sin(kx)\left[B_{k+} + B_{k-}\right] = 100. \leftrightarrow B_{k+} + B_{k-} = 100,$$

$$B_{k+}e^{ky} + B_{k-}e^{-ky} = 100\frac{e^{k(L-y)} - e^{k(y-L)}}{e^{kL} - e^{-kL}} = 100\frac{\sinh(k(L-y))}{\sinh kL}$$

Now, we determine the A_k coefficients by Fourier's trick,

$$A_k = \frac{2}{L}\int_0^L \sin\left[\frac{n\pi x}{L}\right] = \frac{4}{n\pi}.$$

Some algebra combines these two equations for the remaining coefficients as

$$T = \sum_n \frac{400\sin(n\pi x/L)}{n\pi}\frac{\sinh(n\pi(L-y)/L)}{\sinh n\pi}.$$

Evaluating at the center of the plate gives

$$T = \sum_{\text{odd } n} \frac{400\sin(n\pi/2)}{n\pi}\frac{\sinh(n\pi/2)}{\sinh n\pi} = 25.3716 - 0.381232 + 0.00988551 + \cdots = 25.$$

3 Some Heat Walks into a Bar

Problem 13.3.2 of Boas (2006). You do not need to make a computer plot of your results.

A bar 10 cm long with insulated sides is initially at 100°. Starting at $t = 0$, the ends are held at 0°. Find the temperature distribution in the bar at time t. (The answer, but not the solution method, is written out in Boas).

As the sides are insulated, we do not expect any dependence on y, i.e., the temperature will be uniform in stripes across the bar, but we do expect dependences on x, t. Thus,

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial T}{\partial t}, \tag{5}$$

$$T = X_k(x)T_k(t), \tag{6}$$

$$\frac{\frac{\partial^2 X_k}{\partial x^2}}{X_k} = \frac{1}{\alpha^2}\frac{\frac{\partial T_k}{\partial t}}{T} = -k^2, \tag{7}$$

$$X_k = A_k \sin(kx) + B_k \cos(kx), \tag{8}$$

$$T_k = e^{-\alpha^2 k^2 t}. \tag{9}$$

The boundary conditions are $T = 0$ after $t = 0$, so we can consider only a sine series, with $k = n\pi/L$.

$$T = \sum_k A_k \sin(kx) e^{-\alpha^2 k^2 t}. \tag{10}$$

At $t = 0$, we can evaluate all of the A_k,

$$A_k = \frac{2}{L}\int_0^L 100 \sin\left[\frac{n\pi x}{L}\right] = \frac{400}{n\pi}. \tag{11}$$

For odd n. Thus,

$$T = \sum_{\text{odd } n} \frac{400 \sin(n\pi x/L)}{n\pi} e^{-(n\pi\alpha/L)^2 t}.$$

4 Earth to Helmholtz

Write the Helmholtz equation in earth coordinates (i.e., interpret the meaning of the Laplacian in these coordinates). Do you think the separable solutions will be the same or different from those found in spherical coordinates? Why or why not? (Hint: think about where you would conveniently apply the boundary conditions as a function of the coordinates in each system).

To interpret the Helmholtz equation in earth coordinates, we just need to look up the Laplacian from (**??**). I'll use u for the Helmholtz unknown, so as not to confuse with longituyde ϕ.

$$\nabla^2 u + k^2 u = 0,$$

$$\frac{1}{(\mathfrak{z} + r_0)^2}\frac{\partial}{\partial \mathfrak{z}}\left((\mathfrak{z} + r_0)^2 \frac{\partial u}{\partial \mathfrak{z}}\right) + \frac{1}{(\mathfrak{z} + r_0)^2 \cos^2\vartheta}\frac{\partial^2 u}{\partial \phi^2} + \frac{1}{(\mathfrak{z} + r_0)^2 \cos\vartheta}\frac{\partial}{\partial \vartheta}\left(\cos\vartheta \frac{\partial u}{\partial \vartheta}\right) + k^2 u = 0.$$

If we assume separable form for u, then

$$u = \sum Z(\mathfrak{z})P(\phi)T(\vartheta)$$

Each term in that series is a product of a function of height ($\mathfrak{z}$) times a function of longitude (ϕ) times a function of latitude (ϑ). Contrast this against the separable solution in spherical coordinates

$$u = \sum R(r)\Phi(\phi)\Theta(\theta)$$

Since there is a simple mapping from spherical to earth coordinates, $r \to r_0 + \mathfrak{z}, \theta \to \vartheta - \pi/2$, which preserves a one-to-one functional relationship between one earth coordinate variable and one spherical coordinate variable, all of the coordinate surfaces are shared between earth coordinates and spherical coordinates. Since separable solutions are solutions where each variable can vary independently, and since the different variables are not mixed up between earth and spherical coordinates, we expect that there will be equivalent functions in earth coordinates (earth harmonics?) to the separable solutions in spherical coordinates (spherical harmonics). The only difference between these functions will be that the arguments will be processed to make the transformation $r \to r_0 + \mathfrak{z}, \theta \to \vartheta - \pi/2$.

5 Wavy Gravy

The equations of motion for linear, irrotational ($\nabla \times \mathbf{u} = 0$) waves in a uniform depth fluid are, where η is the displacement of the surface from $z = 0$, ϕ is the velocity potential so that derivatives are the horizontal and vertical velocity ($\frac{\partial \phi}{\partial x} = u$, $\frac{\partial \phi}{\partial z} = w$), D is the depth, and g is the gravitational acceleration ($9.81\,\mathrm{m\,s^{-2}}$).

$$\nabla^2 \phi = 0 \tag{12}$$

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z} \qquad \text{at } z = 0 \tag{13}$$

$$\frac{\partial \phi}{\partial t} = -g\eta \qquad \text{at } z = 0 \tag{14}$$

$$\frac{\partial \phi}{\partial z} = 0 \qquad \text{at } z = -D \tag{15}$$

(Part a) is the example problem already solved above in example 11.6. You don't need to do it again, begin with part b. I repeat it here only to point out that part b builds on this result.) Assume a normal mode/plane wave form for $\phi(x, y, z, t)$ and $\eta(x, y, t)$ (note that η is at the surface only, so does not depend on z). Show that the condition for nontrivial solutions (remember how to avoid applying Cramer's rule–set the determinant to zero!) is $\sigma^2 = g\kappa \tanh \kappa D$, where σ is the frequency in time of each mode and $\kappa = \sqrt{k^2 + l^2}$ is wavenumber of the wave, and k and l are the wavenumbers in the x and y directions.

b) In very deep water, $\sigma^2 = g\kappa \tanh \kappa D \approx g\kappa$. Redo the normal mode analysis in a), but in the semi-infinite plane (i.e., infinite depth) ocean.

c) Compare the deep water case in b) to the nondispersive waves (c is *not* a function of wavenumber κ in nondispersive waves) of previous problems. Show that the phase speed of deep water waves ($c = \sigma/\kappa$ where $\kappa D \gg 1$) propagates at different speeds depending on the wavenumber, whereas nondispersive, shallow-water waves ($c = \sigma/\kappa$ where σ is the shallow water limit where $\kappa D \ll 1$) always have the same speed c regardless of κ.

b) Instead of considering both roots for m, if the ocean is infinitely deep we disregard the exponentially growing solution $e^{\kappa D}$, so that $P_- = 0$ and

$$\begin{aligned} i\sigma E &= \kappa P_+ \\ i\sigma P_+ &= -gE \\ \kappa P_+ e^{-\kappa D} &= 0 \end{aligned}$$

The last equation is satisfied if $D = \infty$, and setting the determinant equal to zero in the first two gives

$$\sigma^2 = g\kappa$$

c) Consider the solutions to b), $\eta = Ee^{ikx}e^{ily}e^{\pm i\sqrt{g\kappa}t}$. If we compare this to the form for waves propagating in x only, $x - ct$, we see that the function that plays the role of $x - ct$ is $kx \pm \sqrt{g\kappa}t$. Thus, $c = \pm\sqrt{g\kappa}/k$, which depends on κ and k, unlike the constant c from the nondispersive theory.

6 Helmholtz in Different Views

Look at `http://tinyurl.com/mljujml` and `http://tinyurl.com/ol3al47`. a) Contrast these against the separation of variables in the Cartesian coordinate cases. b) Why aren't the solutions sines and cosines? c) How can it matter which coordinate system we choose–that is, what is so special about separable solutions?

a) The separable solutions to the Helmholtz equation in spherical and cylindrical coordinates differ from each other and from Cartesian coordinates. In Cartesian coordinates, we have sines, cosines, and exponentials (or equivalently sinh and cosh). In parabolic cylindrical coordinates, the z axis has sines and cosines, as in Cartesian, but the u and v separable solutions are very different because of the Weber differential equations that result during separation. In the spherical case, θ has sines and cosines as $\frac{\partial^2}{\partial\phi^2}$ occurs without variable coefficients in the spherical version of Helmholtz, but both r and ϕ have non-constant coefficients in that system, so different separable solutions result–the Legendre polynomials (and the associated Legendre functions when r is retained).

b) The other functions that differ from sines and cosines occur because in the new coordinate systems, the coefficients of the derivatives are not constants. We know that a constant coefficient, homogeneous differential equation will have exponentials, sines, and cosines, but in these other coordinate systems the coefficients are not constant. Physically, these differences result from the curvature of the coordinate surfaces in space.

c) The overall solution to the differential equation is a *sum* over all separable solutions, such that any initial or boundary conditions supplied are satisfied. However, each term in the separable series can depend on the coordinate system, as they are found by the property that they are constant over different surfaces (spherical shells, cylinders, etc.). They are all orthogonal function sets, and in fact they are generally complete, so that we can convert between different representations in different coordinate systems, but only when the whole sum is retained. Each term in the series is not guaranteed to be represented by a single term in the series in a different coordinate system (just like our Taylor series versus Fourier series problem on the midterm).

References

Boas, M. L., 2006. Mathematical methods in the physical sciences, 3rd Edition. Wiley, Hoboken, NJ.