Fall 2014 GEOL0350–GeoMath Final Exam v1

1 ODElay

We have already found the general solution to an ordinary differential equation for seismic wave displacements (x, y) away from their steady positions (x^*, y^*) at a given location and it is (where c_1, c_2, c_3 may be complex constants):

$$\begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{(i-0.1)t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{(-i-0.1)t} + c_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-10t}.$$

1.1 One Behavior

If $(x - x^*, y - y^*) = (1, 1)$ and $\frac{\mathrm{d}(x-x^*, y-y^*)}{\mathrm{d}t} = (-10, -10)$ at $t = 0$, what happens at larger t?

1.2 Another Behavior

What if $(x - x^*, y - y^*) = (-2, 2)$ at $t = 0$, what kinds of motion result?

1.3 Quandary

How does one distinguish c_1 from c_2 since both describe initial positions along (1,-1)?

1.4 Fixed Point

What is the long time behavior of this system, regardless of initial conditions?

1.5 System

What category of ordinary differential equation would you guess was solved by this solution?

2 PDQ PDE

Consider two functions that satisfy the two-dimensional Poisson equation and two that satisfy Helmholtz's equation within the two-dimensional domain $-1 \leq x \leq 1, -1 \leq y \leq 1$:

$$\nabla^2\phi = f(x,y), \qquad \nabla^2\psi = f(x,y),$$
$$\nabla^2\alpha + k^2\alpha = 0, \qquad \nabla^2\beta + k^2\beta = 0.$$

2.1 Super 1

Use superposition to determine an equation that $\phi - \psi$ solves.

2.2 Super 2

Now determine an equation that $\alpha - \beta$ solves.

2.3 BCs

Suppose ϕ and ψ both satisfy the following boundary conditions: $\psi = \phi = 0$ on $x = \pm 1$, and $\psi = \phi = 1$ on $y = \pm 1$ in addition to their field equations. Now what equations does $\phi - \psi$ solve, and what is the solution for $\phi - \psi$?

2.4 BCs 2

Suppose ϕ and ψ both satisfy the following boundary conditions: $\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial x} = 0$ on $x = \pm 1$, and $\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial y} = 0$ on $y = \pm 1$ in addition to their field equations. Now what equations does $\phi - \psi$ solve, and what is the solution?

3 Absolut Dynamical Systems

Consider the following dynamical system, which is extremely nonlinear but only at one point.

$$\dot{x} = r - |x| \tag{1}$$

3.1 Sketch $\dot{x}$ versus x for $r < 0$, $r = 0$, $r > 0$ and denote stable and unstable fixed points.

3.2 Sketch x^* versus r.

3.3 What kind of bifurcation is exhibited by (1)?

3.4 Contrast the bifurcation in system (1) versus the one in (2).

$$\dot{x} = r(1 - |x|) \tag{2}$$

4 Stats–Whole Lotta Shakin

The Gutenberg-Richter Law gives the number N of earthquakes in a region of time period of at least magnitude M. It can be expressed under simple conditions over a time window and location where there are 100 earthquakes of any magnitude as:

$$N = 10^{2-M}$$

A useful hint in this question will be:

$$\int^{x} 10^{2-M}\,\mathrm{d}M = -\frac{10^{2-x}}{\log 10}$$

Also note that M ranges from 0 to ∞.

4.1 Normalization

The probability density function $\rho(M)$ is proportional to the number of events in this case, but is normalized differently, so $\rho(M) = C10^{2-M}$. Use the integral formula above to find C and express the probability density function for the Gutenberg-Richter law.

4.2 Likelihood

How likely is a magnitude 5 or greater event versus a magnitude 4 or greater event?

4.3 Percentile

What magnitude is the 99th percentile?

4.4 Hypothesis

With significance $p < 0.01$, if only one earthquake was observed, what magnitude or greater would be needed to reject the hypothesis that this Gutenberg-Richter Law applies?

5 Stats–Central Limit & Monte Carlo

A new mode of climate variability is detected with a timescale shorter than a month, and it involves three different locations of the atmospheric jet stream. They are not of equal likelihood, but they are equally far from one another in distance based on a unit L. By looking back at historical records, it is determined that $y = 1L$ occurs 250 out of 2500 months, $y = 2L$ occurs 500 out of 2500 months, $y = 3L$ occurs 750 out of 2500 months, and $y = 4L$ occurs 1000 out of 2500 months. No other locations are observed.

5.1 Draw

Draw a histogram of the historical record data.

5.2 Mean

What is the average position y of the jet stream?

5.3 Variance

What is the variance of the position?

5.4 Combine 2

If 2 independent months are chosen, what is the likelihood that at least 1 of them will have a jet stream at $y = L$?

5.5 Mean of 4

If 4 independent months are chosen, use the central limit theorem to estimate the likelihood that the mean of these 4 months will be lower than $2L$.

5.6 Bootstrapping

How does one use bootstrapping to improve this estimate of a particular 4 month sample? What about uncertainties on other statistics of the distribution not subject to the central limit theorem (e.g., skewness, kurtosis)?

5.7 Monte Carlo

Since we have a larger set of data than just 4 months from the historical data, how might Monte Carlo methods be used to make an even better estimate of the statistics of *any* 4 month's mean and uncertainty in *any* 4 month's samples of skewness and kurtosis drawn from the historical distribution?