Fall 2016 GEOL0350–GeoMath Final Exam

1 Wavy Gravy

The seismic wave equation is:

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = \left(\frac{\lambda}{\rho} + 2\beta^2\right) \nabla(\nabla \cdot \mathbf{u}) - \beta^2 \nabla \times \nabla \times \mathbf{u}. \tag{1}$$

Where $\mathbf{u}$ is the displacement vector field, and λ and β are measures of elasticity (Lamé parameters). If we take the curl and divergence of this equation, we find, respectively,

$$\frac{\partial^2 \nabla \times \mathbf{u}}{\partial t^2} = \beta^2 \nabla^2 \left(\nabla \times \mathbf{u}\right), \qquad \frac{\partial^2 \nabla \cdot \mathbf{u}}{\partial t^2} = \left(\frac{\lambda}{\rho} + 2\beta^2\right) \nabla^2 \left(\nabla \cdot \mathbf{u}\right). \tag{2}$$

These two equations are wave equations for the curl of the displacement (S-waves) and the divergence of the displacement (P-waves). You can assume λ, ρ, β are positive.

1.1 Diverge and Separate

Use the method of separation of variables to separate out a time dependent part of the displacement divergence ($T_{\cdot}(t)$) from a space dependent part ($X_{\cdot}(x, y, z)$). What equation does each $X(x, y, z)$ solve? (You don't have to split up the spatial coordinates.)

1.2 Curl and Separate

Use the method of separation of variables to separate out a time dependent part of these equations ($T_\times(t)$) from a space dependent part ($X_\times(x, y, z)$).

1.3 Guess and Check: Wavenumber and Frequency

Assume that both the divergence and the curl have $T(t) \propto e^{i\omega t}$ with frequency ω, and $X(x, y, z) \propto e^{\pm ikx}$ with wavenumber $\pm k$ (here we are considering only one spatial direction for simplicity). Use (2) to write two equations relating k to ω (these are called the dispersion relations).

1.4 Phase Speed

The *phase speed* of the waves is $c = \omega/k$. Find the phase speed for the divergence and curl of the displacement. Which has the faster phase speed?

1.4.1 Initiate One or the Other

A seismic disturbance $\mathbf{u}_0$ will generally trigger both divergence $\nabla \cdot \mathbf{u}_0$ and curl $\nabla \times \mathbf{u}_0$ initial conditions that will propagate away as waves. What kind of mathematical form for $\mathbf{u}_0$ would possess only divergence waves? What kind of mathematical form for $\mathbf{u}_0$ would have only curl waves?

2 PDF PDQ

Three different probability distribution functions are being considered to explain the results of an experiment, which has independent measurements of x repeatedly. They are:

$$\rho_u(x;\sigma,\mu) = \frac{\mathcal{H}(\sqrt{3}\sigma - |x-\mu|)}{2\sqrt{3}\sigma}, \quad \rho_e(x;\sigma,\mu) = \frac{1}{\sigma\sqrt{2}} e^{-\sqrt{2}|x-\mu|/\sigma}, \quad \rho_n(x;\sigma,\mu) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)} \tag{12}$$

Where $\mathcal{H}$ is the Heaviside function (zero when it's argument is negative and 1 when positive). The three distributions have the same mean μ and standard deviation σ. Half of the experimental data fall above the mean and half below. Here we consider only the half that are above the mean.

Hints:

$$\int_{\mu}^{\infty} \rho_u(x;\sigma,\mu)\,\mathrm{d}x = 0.5, \quad \int_{\mu}^{\infty} \rho_e(x;\sigma,\mu)\,\mathrm{d}x = 0.5, \quad \int_{\mu}^{\infty} \rho_n(x;\sigma,\mu)\,\mathrm{d}x = 0.5 \tag{13}$$

$$\int_{\mu+\sigma}^{\infty} \rho_u(x;\sigma,\mu)\,\mathrm{d}x = 0.21, \quad \int_{\mu+\sigma}^{\infty} \rho_e(x;\sigma,\mu)\,\mathrm{d}x \approx 0.12, \quad \int_{\mu+\sigma}^{\infty} \rho_n(x;\sigma,\mu)\,\mathrm{d}x \approx 0.16, \tag{14}$$

$$\int_{\mu+2\sigma}^{\infty} \rho_u(x;\sigma,\mu)\,\mathrm{d}x = 0, \quad \int_{\mu+2\sigma}^{\infty} \rho_e(x;\sigma,\mu)\,\mathrm{d}x \approx 0.03, \quad \int_{\mu+2\sigma}^{\infty} \rho_n(x;\sigma,\mu)\,\mathrm{d}x \approx 0.02, \tag{15}$$

$$\int_{\mu+3\sigma}^{\infty} \rho_u(x;\sigma,\mu)\,\mathrm{d}x = 0, \quad \int_{\mu+3\sigma}^{\infty} \rho_e(x;\sigma,\mu)\,\mathrm{d}x \approx 0.007, \quad \int_{\mu+3\sigma}^{\infty} \rho_n(x;\sigma,\mu)\,\mathrm{d}x \approx 0.001. \tag{16}$$

2.1 Flatlining

Suppose one datum falls between μ and $\mu+\sigma$ and one datum falls between $\mu+\sigma$ and $\mu+2\sigma$. Which distribution is closest to the histogram?

2.2 Gambler's Fallacy

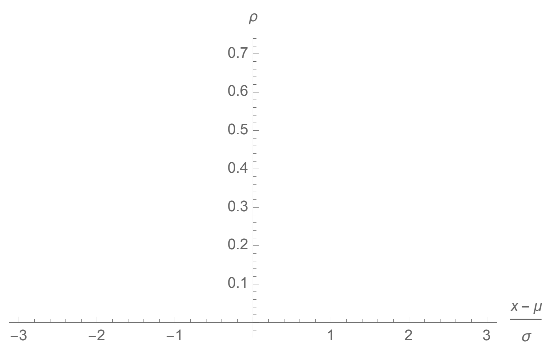
If the true distribution were normal (ρ_n), there are roughly twice as many data expected to fall between μ and $\mu+\sigma$ as between $\mu+\sigma$ and ∞. Suppose the first two experiments out of three fall between μ and $\mu+\sigma$. Which is the most likely next range: between μ and $\mu+\sigma$ or $\mu+\sigma$ and ∞?

2.3 Combine the Probabilities

After two experiments, we decide to evaluate the following likelihood, based on the two results ($x > \mu+2\sigma, x > \mu$). If p(A) is the probability of experiment 1 being $x > \mu+2\sigma$, and p(B) is the probability of experiment 2 being $x > \mu$, what are the probabilities of any of the two $p(A+B)$ being true and both of the two $p(AB)$, *for each of the three distributions*? Can you eliminate any of the distributions by these results?

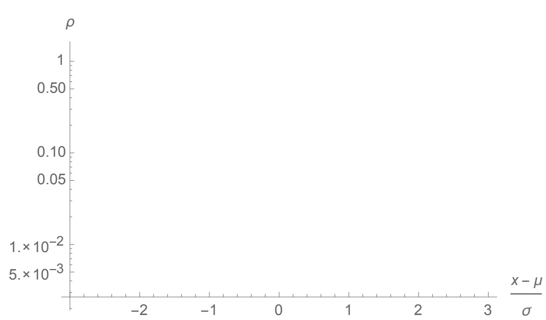
2.4 Graph em linear!

Plot the probability distributions given between $-3\sigma + \mu$ and $+3\sigma + \mu$ on the following (linear) axes.



2.5 Graph em log!

Plot the probability distributions given between $-3\sigma+\mu$ and $+3\sigma+\mu$ on the following (semi-log) axes.



3 Erosion by a Pebble in a Basin

Over time, a basin has formed with the shape (polar coordinates r, θ, dimensionless units):

$$h(r) = r^4 - 2r^2 + 1. \tag{22}$$

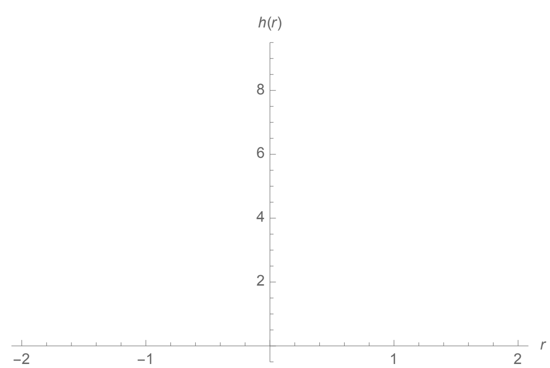
This basin has formed like this because there is a pebble in the basin, which is driven by the wind to have the following dynamics,

$$\dot{r} = -\nabla h(r) = -4r^3 + 4r^2, \tag{23}$$
$$\dot{\theta} = 1. \tag{24}$$

(Note that negative values of r are used, so that a complete cross-section can be drawn, but of course these are just equivalent to shifting θ by π.)

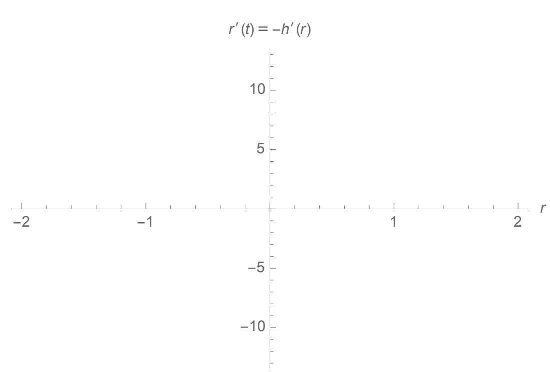
3.1 Fill in the hole!

Plot the shape of the basin $h(r)$.



3.2 Slope of the hole!

Plot $\dot{r}$.



3.3 Steady r

Show on the preceding plots of $h(r)$ and $\dot{r}$ where $\dot{r}$=0. Are these stable or unstable to perturbations in r?

3.4 Round and Round

Now examine the $\dot{\theta}$ equation, which describes the motion of the pebble around the axis of the basin. What kind of motion does it describe?

3.5 Attractors

Describe the attractors of this system.

3.6 Angular Momentum?

Does this system conserve angular momentum $r^2\dot{\theta}$?

3.7 Deformed and Deforming

If the motion of the pebble is partly to credit for the shape of the basin, how is this related to the dynamics of the pebble?

4 Under Pressure

The momentum equation for a solid, liquid, or gas is.

$$\rho \frac{D\mathbf{v}}{Dt} = \nabla \cdot \sigma + \rho \mathbf{g}. \tag{25}$$

4.1 Stress Tensors

The difference between a solid, a liquid, and a gas stem from the stress tensor. Write the stress tensor σ_{ij} for each:

4.2 Pressure Only

For a liquid, solid or gas, if there is no velocity ($v = 0$) and no displacement ($\Delta x = 0$), there is only one balance possible in the momentum equation. What is it?

4.3 Solid Balance

If a solid is motionless ($v = 0$), but there has been a displacment ($\Delta x \neq 0$), is it possible for a solid to have a different steady balance than the previous one?

4.4 Fluid Balance

If a fluid is motionless ($v = 0$), but there has been a displacment ($\Delta x \neq 0$), is it possible for a fluid to have a different steady balance than the previous one?

4.5 Once again into the Breach

Mathematically, why can you separate the part of the stress tensor that depends on the pressure from the rest of the stress tensor that depends on displacement or velocity?