Assignment 2 for GEOL 1820: Geophysical Fluid Dynamics, Waves and Mean Flows Edition Due Oct. 17, 2016

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Getting Help!

I am usually available by email. You can make an appointment other times. Just check my calendar at http://fox-kemper.com/contact and suggest a time that works for you.

1 Problem 1: Parabolic Ray Focusing

Do problem 3.4.2 of (Bühler, 2014).

We are interested in the intersection of rays arriving from right to left into a reflector at $y^2 = ax$. If we consider one particular y, $y = b$ we can evaluate the tangent and normal directions at the intersection point $(b^2/a, b)$. By symmetry, we can just assume $b > 0$. We begin with a Taylor series about this point, then truncate to the linear tangent function. Taking a step of length Δx along this direction yields,

$$
\begin{aligned}
y &= b + \frac{a}{2b}\left(x - \frac{b^2}{a}\right) - \frac{a^2}{8b^3}\left(x - \frac{b^2}{a}\right)^2 + \dots \\
y &\approx b + \frac{a}{2b}\left(x - \frac{b^2}{a}\right) \\
\Delta y &= \frac{a}{2b}\Delta x.
\end{aligned}
$$

Thus, the tangent and normal unit vectors are given by,

$$\hat{\mathbf{t}} = \frac{(\Delta x, \Delta y)}{\sqrt{\Delta x^2 + \Delta y^2}},$$
$$\hat{\mathbf{n}} = \frac{(\Delta y, -\Delta x)}{\sqrt{\Delta x^2 + \Delta y^2}},$$

Examining the figure, we see that the slope of the outgoing ray is,

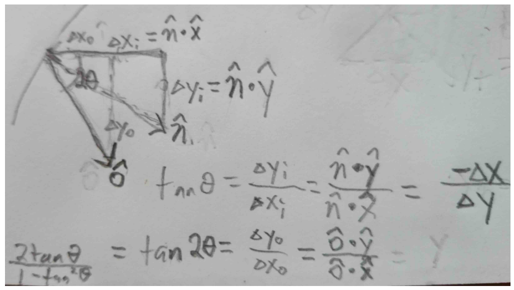

Figure 1: Sorting out the incoming and outgoing triangles.

$$\frac{\Delta y_o}{\Delta x_o} = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{-2\frac{\Delta x}{\Delta y}}{1-(\frac{\Delta x}{\Delta y})^2} = \frac{2\frac{\Delta y}{\Delta x}}{1-(\frac{\Delta y}{\Delta x})^2} = \frac{\frac{a}{b}}{1-(\frac{a}{2b})^2} = \frac{4ab}{4b^2-a^2}$$

Adding constants to ensure that the intercept hits $y = b$ when $x = \frac{b^2}{a}$, we find

$$y = b + \left(x - \frac{b^2}{a}\right)\frac{4ab}{4b^2 - a^2}.$$

2 Problem 2: Reconciling Wave Flux with Energy and Group Velocity

In the WKB expansion of (Bühler, 2014), we have the ansatz (i.e., solution guess) and equations,

$$h = A(x,y)e^{i(\kappa_0 s(x,y) - \alpha - \omega t)} \tag{1}$$
$$|\nabla s|^2 = n^2, \tag{2}$$
$$\nabla \cdot \left(\frac{A^2}{n^2}\nabla s\right) = 0. \tag{3}$$

In the phase velocity, group velocity slides (see also Chp. 1 of Chapman & Rizzoli, 1989)), it is discussed how energy propagates with the group velocity.

$$\mathbf{c}_g = \frac{\partial \omega}{\partial k}\hat{\mathbf{k}} \tag{4}$$

Relate these two frameworks to express the argument of the divergence in (3) as a group velocity times a conserved quantity. What is the conserved quantity? How do you relate $\frac{\partial \omega}{\partial k}$ to $s(x,y)$? Hints: Here ω doesn't vary k does, and $c_g = c_p$ for shallow water waves.

We note in the shallow water equations that $n^2 = c_0^2/c_g^2$. Thus, the shallow water flux rule can be rewritten as, using this relationship and the eikonal equation:

$$0 = \nabla \cdot \left(\frac{A^2 \nabla s}{n^2} \right) = \nabla \cdot \left(\frac{A^2 \nabla s}{|n||\nabla s|} \right) = \nabla \cdot \left(\frac{A^2 \mathbf{c}_g}{|c_0|} \right) \tag{5}$$

As c_0 is constant by assumption, this is equivalent to a flux of amplitude squared

$$0 = \nabla \cdot \left(g A^2 \mathbf{c}_g \right), \tag{6}$$

or a flux of wave action (because ω is spatially constant)

$$0 = \nabla \cdot \left(\frac{g A^2 \mathbf{c}_g}{\omega} \right). \tag{7}$$

Using (2.29) we can see that this amplitude squared resembles the phase-averaged energy of a plane wave, if equipartition applies (as it does in the nonrotating, shallow water case).

$$\overline{E} = \frac{H}{2} \overline{u_w'^2 + v_w'^2} + \frac{g}{2} \overline{h_w'^2} = g \overline{h_w'^2} \tag{8}$$

3 Comparison of Stationary Phase

Compare section 3.2.3 of Bühler (2014) with section 1.4 of Chapman & Rizzoli (1989). What is the essence of the method of stationary phase?

The equation in section 3.2.3 of Bühler (2014) treated with the method of stationary phase is

$$\hat{h}(x) = i\kappa_0 \beta \int_{\mathcal{C}} \frac{\cos\alpha + 1}{\sqrt{\kappa_0 r(\sigma)}} \exp(i\kappa_0 r(\sigma)) \mathrm{d}\sigma + \mathrm{O}(\kappa_0^{-1}).$$

In section 1.4 of Chapman & Rizzoli (1989) the equation treated is

$$\phi(x,t) = \int_{-\infty}^{\infty} A(k) \exp(ikx - \Omega(k)t) \mathrm{d}k.$$

In both cases, the integral in question takes the form

$$I(x) = \int_a^b f(t) \exp(ix\phi(t)) \mathrm{d}t.$$

If the argument of the exponent varies rapidly, then the integral goes toward zero. For the general method, I will excerpt from (Bender & Orszag, 1999) which gives additional detail, but a successful answer to this question need not be this detailed. Using integration by parts it can be shown that this integral approaches

$$I(x) \sim \frac{f(t)}{ix\phi'(t)} e^{ix\phi(t)} \Big|_{t=a}^{t=b}, \qquad x \to +\infty$$

Unless this formula breaks down, which occurs if $\phi'(t) = 0$, i.e., a stationary point. More than one stationary point may occur within the bounds of integration, and the integral may be approximated by a sum over all of them. Suppose there is just one stationary point, and expand the integrand in a Taylor series about that point and an integral centered on that region, then

$$I(x) \sim \int_a^{a+\epsilon} f(a) \exp\left[ix\left(\psi(a) + \frac{1}{p!}\psi^{(p)}(a)(t-a)^p\right)\right] \mathrm{d}t, \qquad x \to \infty.$$

Thus, if $f(a)$ doesn't vanish but $\psi'(a) = 0$, then the leading non-vanishing term of the Taylor series dominates. In the cases in the books, the second derivative is assumed not to vanish, which amounts to

$$I(x) \sim f(a) e^{ix\phi(a) + i\pi \mathrm{sgn}(\phi''(a))/4} \sqrt{\frac{\pi}{2x|\phi''(a)|}}, \qquad x \to \infty.$$

For your interest, if you have to continue to higher order terms in the Taylor series, then the integral is approximated by the first nonvanishing term by

$$I(x) \sim f(a) e^{ix\phi(a) + i\pi \mathrm{sgn}(\phi''(a))/(2p)} \left|\frac{p!}{2x|\phi^{(p)}(a)|}\right|^{1/p} \frac{\Gamma(1/p)}{p}, \qquad x \to \infty.$$

4 Phase and Group Velocity

Find the phase speed in k, ℓ directions and the group velocity (vector, gradient w.r.t. k, ℓ) for the following dispersion relations for x wavenumber k and y wavenumber ℓ. $(k, l) = \boldsymbol{\kappa}$, and $\kappa = |\boldsymbol{\kappa}|$. Subscript 0 indicates a constant. Hint: $\frac{\partial \kappa}{\partial k} = \frac{k}{\kappa}, \frac{\partial \kappa}{\partial \ell} = \frac{\ell}{\kappa}$

$$\omega = \mathbf{c}_0 \cdot \boldsymbol{\kappa} \tag{9}$$

$$\omega^2 = \mathbf{c}_0^2\ \kappa^2 \tag{10}$$

$$\omega^2 = gH\kappa^2 \quad \text{shallow-water waves} \tag{11}$$

$$\omega^2 = g\kappa \quad \text{deep-water waves} \tag{12}$$

$$\omega = \frac{-\beta k}{\kappa^2} \quad \text{Rossby waves} \tag{13}$$

$$\omega = \mathbf{c}_0 \cdot \boldsymbol{\kappa}$$
$$\frac{\omega}{k} = \frac{\mathbf{c}_0 \cdot \kappa}{k} = \mathbf{c}_0 \cdot \hat{x} + \mathbf{c}_0 \cdot \hat{y}\frac{l}{k}$$
$$\frac{\omega}{l} = \frac{\mathbf{c}_0 \cdot \kappa}{l} = \mathbf{c}_0 \cdot \hat{x}\frac{k}{l} + \mathbf{c}_0 \cdot \hat{y}$$
$$\mathbf{c}_g = \nabla_\kappa \omega = (\mathbf{c}_0 \cdot \hat{x}, \mathbf{c}_0 \cdot \hat{y})$$

Next:

$$\omega^2 = \mathbf{c}_0^2\ \kappa^2 = c_0^2 \boldsymbol{\kappa} \cdot \boldsymbol{\kappa}, \qquad \omega = \pm c_0 \kappa$$
$$\frac{\omega}{k} = \frac{\pm c_0 \kappa}{k}$$
$$\frac{\omega}{l} = \frac{\pm c_0 \kappa}{l}$$
$$\mathbf{c}_g = \nabla_\kappa \omega = \frac{c_0}{\kappa}(k, l)$$

Note how the velocity in this form has to be isotropic! Next:

$$\omega^2 = gH\kappa^2 = gH\boldsymbol{\kappa}\cdot\boldsymbol{\kappa}, \qquad \omega = \pm\sqrt{gH}\kappa$$
$$\frac{\omega}{k} = \frac{\pm\sqrt{gH}\kappa}{k}$$
$$\frac{\omega}{l} = \frac{\pm\sqrt{gH}\kappa}{l}$$
$$\mathbf{c}_g = \nabla_\kappa\omega = \frac{\sqrt{gH}}{\kappa}(k,l)$$

Note that the phase speed in these directions is *not* equal to the group velocity, unless k or l is zero. That is, the phase speed and group speed are equal only in the direction of propagation. Next:

$$\omega^2 = g\kappa, \qquad \omega = \pm\sqrt{g\kappa}$$
$$\frac{\omega}{k} = \frac{\pm\sqrt{g\kappa}}{k}$$
$$\frac{\omega}{l} = \frac{\pm\sqrt{g\kappa}}{l}$$
$$\mathbf{c}_g = \nabla_\kappa\omega = \frac{\sqrt{g}}{2\kappa^{3/2}}(k,l)$$

Again the phase speed in these directions is *not* equal–or even proportional–to the group velocity, unless k or l is zero. In the direction of propagation, the phase speed is twice the group speed. Next,

$$\omega = \frac{-\beta k}{\kappa^2}$$
$$\frac{\omega}{k} = \frac{-\beta}{\kappa^2}$$
$$\frac{\omega}{l} = \frac{-\beta k}{l\kappa^2}$$
$$\mathbf{c}_g = \nabla_\kappa\omega = \frac{\beta}{\kappa^4}\left(k^2 - l^2, 2kl\right)$$

There is very little relationship at all between phase speeds and group velocities in this case, which is the short Rossby wave limit. The more complete dispersion relation for Rossby waves (which has a deformation radius factor) does have a non-dispersive long wave limit.

5 Piecewise Beach

Consider the following index of refraction variations:

$$n(x,y) = \begin{cases} n_d & x \le -L \\ x\frac{n_s - n_d}{2L} + \frac{n_s+n_d}{2} & |x| \le |L| \\ n_s & x \ge L \end{cases} \tag{14}$$

5.1

For shallow water waves, how does the depth vary over this region?

Substituting $n = c_0/\sqrt{gH}$:

$$\frac{c_0}{\sqrt{gH(x,y)}} = \begin{cases} \frac{c_0}{\sqrt{gH_d}} & x \le -L \\ \frac{x}{2L}\left(\frac{c_0}{\sqrt{gH_s}} - \frac{c_0}{\sqrt{gH_d}}\right) + \frac{1}{2}\left(\frac{c_0}{\sqrt{gH_s}} + \frac{c_0}{\sqrt{gH_d}}\right) & |x| \le |L| \\ \frac{c_0}{\sqrt{gH_s}} & x \ge L \end{cases}$$

Canceling common factors,

$$\frac{1}{\sqrt{H(x,y)}} = \begin{cases} \frac{1}{\sqrt{H_d}} & x \leq -L \\ \frac{x}{2L}\left(\frac{1}{\sqrt{H_s}} - \frac{1}{\sqrt{H_d}}\right) + \frac{1}{2}\left(\frac{1}{\sqrt{H_s}} + \frac{1}{\sqrt{H_d}}\right) & |x| \leq |L| \\ \frac{1}{\sqrt{H_s}} & x \geq L \end{cases}$$

Thus, a linear variation in n indicates a variation in depth like this:

$$H(x,y) = \begin{cases} H_d & x \leq -L \\ \frac{4L^2}{x^2}\left(\frac{H_s H_d}{H_d + H_s - 2\sqrt{H_s H_d}}\right) + \frac{8L}{x}\left(\frac{H_s H_d}{H_d - H_s}\right) + 4\left(\frac{H_s H_d}{H_d + H_s + 2\sqrt{H_s H_d}}\right) & |x| \leq |L| \\ H_s & x \geq L \end{cases}$$

Note that if $H_s = H_d$, then the two terms proportional to x/L do not appear (as opposed to being infinite which is how they appear here).

5.2

For waves that are directly incident on the slope (i.e, $s(x,y) \propto x$ and $A = 1$ at $x \ll -L$), solve for $A(x,y)$ and $s(x,y)$ using equations (3.6) and (3.8) of Bühler (2014).[1]

For steady directly incident waves, the governing PDEs become ODEs. They are:

$$s_x^2 = n^2$$
$$\frac{d}{dx}\left(\frac{A^2}{n^2}s_x\right) = 0$$
$$\frac{dx}{d\tau} = s_x$$
$$\frac{ds_x}{d\tau} = nn_x = \begin{cases} 0 & x \leq -L \\ \frac{n_s - n_d}{2L} & |x| \leq |L| \\ 0 & x \geq L \end{cases}$$

The equations that are no longer needed are:

$$\frac{dy}{d\tau} = s_y = 0$$
$$\frac{ds_y}{d\tau} = nn_y = 0$$

because these just indicate that s is constant in y (crests are parallel with topography) and independent of τ (i.e., increments in τ measure distance along the rays which are oriented in x).
Consider solving just the first two equations (those that preceded the formal method of characteristics),

[1] Typo: Sorry! s should have been proportional to x instead of equal to x.

combining them, we find

$$\frac{d}{dx}\left(\pm\frac{A^2}{n}\right) = 0$$
$$\frac{A^2}{n} = \text{constant} = \frac{1}{n_d}$$
$$\frac{A^2(x,y)}{n(x,y)} = \frac{1}{n_d} = \begin{cases} \frac{A^2}{n_d} & x \leq -L \\ \frac{A^2}{x\frac{n_s-n_d}{2L}+\frac{n_s+n_d}{2}} & |x| \leq |L| \\ \frac{A^2}{n_s} & x \geq L \end{cases}$$
$$A^2 = \begin{cases} 1 & x \leq -L \\ \frac{x}{2L}\left(\frac{n_s}{n_d}-1\right)+\frac{1}{2}\left(\frac{n_s}{n_d}+1\right) & |x| \leq |L| \\ \frac{n_s}{n_d} & x \geq L \end{cases}$$

The transition to ODEs means that the method of characteristics is redundant, but we can still do the solution that way as well.

$$\frac{ds_x}{d\tau} = nn_x = \begin{cases} 0 & x \leq -L \\ \frac{n_s-n_d}{2L} & |x| \leq |L| \\ 0 & x \geq L \end{cases}$$
$$s_x = \begin{cases} a & x \leq -L \\ \frac{n_s-n_d}{2L}\tau + b & |x| \leq |L| \\ c & x \geq L \end{cases}$$
$$= \begin{cases} n_d & x \leq -L \\ \frac{n_s-n_d}{2L}\tau + b & |x| \leq |L| \\ n_s & x \geq L \end{cases}$$
$$\frac{dx}{d\tau} = s_x = \begin{cases} n_d & x \leq -L \\ \frac{n_s-n_d}{2L}\tau + b & |x| \leq |L| \\ n_s & x \geq L \end{cases}$$

We can use the total derivative of s versus τ to establish the remaining relationships.

$$\frac{ds}{d\tau} = s_x\frac{dx}{d\tau} + \cancelto{0}{s_y\frac{dy}{d\tau}} = s_x^2 + \cancelto{0}{s_y^2} = n^2$$
$$\tau = x$$
$$b = \frac{n_s+n_d}{2}$$

The amplitude can be found as above.

5.3

For waves that hit the slope obliquely (i.e, $s(x,y) \propto (x\cos\theta_0 + y\sin\theta_0)$ and $A = 1$ at $x \ll -L$), solve for $A(x,y)$ and $s(x,y)$ using equations (3.6) and (3.8) of Bühler (2014).[2]

[2]Typo: $s(x,y)$ should be proportional to x not equal!.

This time, we cannot be so quick to convert to an ODE, but things are still pretty easy since

$$\frac{dy}{d\tau} = s_y \propto \sin\theta_0$$
$$\frac{ds_y}{d\tau} = nn_y = 0$$

So, s_y is just fixed throughout the domain (i.e., the waves only contract and expand in x, their y wavelength doesn't change with x).

$$\frac{ds_x}{d\tau} = nn_x = \begin{cases} 0 & x \le -L \\ \frac{n_s - n_d}{2L} & |x| \le |L| \\ 0 & x \ge L \end{cases}$$
$$\frac{dx}{d\tau} = s_x = \begin{cases} a & x \le -L \\ \frac{n_s - n_d}{2L}\tau + b & |x| \le |L| \\ c & x \ge L \end{cases}$$
$$\frac{dy}{d\tau} = s_y = d\sin\theta_0$$

Now, the distance along the ray is a bit more interesting,

$$\frac{ds}{d\tau} = s_x\frac{dx}{d\tau} + s_y\frac{dy}{d\tau} = s_x^2 + s_y^2 = n^2 = \begin{cases} n_d^2 & x \le -L \\ \left(x\frac{n_s - n_d}{2L} + \frac{n_s + n_d}{2}\right)^2 & |x| \le |L| \\ n_s^2 & x \ge L \end{cases}$$
$$= n^2 = \begin{cases} n_d^2\left(\cos^2\theta_0 + \sin^2\theta_0\right) & x \le -L \\ \left(x\frac{n_s - n_d}{2L} + \frac{n_s + n_d}{2}\right)^2 & |x| \le |L| \\ n_s^2 & x \ge L \end{cases}$$
$$s_y^2 = n_d^2\sin^2\theta_0$$
$$s_x^2 = \begin{cases} n_d^2\cos^2\theta_0 & x \le -L \\ \left(x\frac{n_s - n_d}{2L} + \frac{n_s + n_d}{2}\right)^2 - n_d^2\sin^2\theta_0 & |x| \le |L| \\ (n_s^2 - n_d^2) + n_d^2\cos^2\theta_0 & x \ge L \end{cases}$$

OK–now on to the amplitude.

$$\nabla\cdot\frac{A^2}{n^2}\nabla s = 0$$

Examining the preceding, s_y is constant, and n^2 doesn't depend on y. At negative x, A^2 is uniform. Thus, A^2 cannot depend on y. Thus, we arrive at a similar constant in x as last time, but it is more difficult to evaluate since n^2 depends on s_y as well as s_x.

$$\nabla\cdot\frac{A^2}{n^2}\nabla s = \partial_x\frac{A^2}{n^2}s_x + \cancelto{0}{\partial_y\frac{A^2}{n^2}}s_y = \partial_x\frac{A^2}{n^2}s_x = 0$$
$$A^2 = \begin{cases} 1 & x \le -L \\ \frac{\left(x\frac{n_s - n_d}{2L} + \frac{n_s + n_d}{2}\right)^2\cos\theta_0}{n_d\sqrt{\left(x\frac{n_s - n_d}{2L} + \frac{n_s + n_d}{2}\right)^2 - n_d^2\sin^2\theta_0}} & |x| \le |L| \\ \frac{n_s^2\cos\theta_0}{n_d\sqrt{(n_s^2 - n_d^2) + n_d^2\cos^2\theta_0}} & x \ge L \end{cases}$$

References

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