

Assignment 3 for GEOL 1820:
Geophysical Fluid Dynamics,
Waves and Mean Flows Edition
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Getting Help!

I am usually available by email. You can make an appointment other times. Just check my calendar at <http://fox-kemper.com/contact> and suggest a time that works for you.

1 Rossby waves created by sidewall undulations

Do problem 4.7.1 of (Bühler, 2014).

We begin with the (short) Rossby wave dispersion relation for the intrinsic frequency, and the group speeds in x and y it implies.

$$\begin{aligned}\hat{\omega} &= -\beta \frac{k}{k^2 + l^2}, \\ \omega &= Uk - \beta \frac{k}{k^2 + l^2}, \\ \mathbf{c}_g \cdot \hat{\mathbf{x}} &= U - \beta \frac{-k^2 + l^2}{(k^2 + l^2)^2}, \\ \mathbf{c}_g \cdot \hat{\mathbf{y}} &= \beta \frac{2kl}{(k^2 + l^2)^2},\end{aligned}$$

We are interested in a plane wave solution for q' , which solves the (steady) equation satisfying $y_s =$

$a \cos(kx)$.

$$\begin{aligned} H_0 q'_t + H_0 U q'_x + \beta v' &= 0, \\ H_0 U q'_x &= -\beta U \frac{dy_s}{dx} = \beta U a k \sin(kx). \end{aligned}$$

Since the boundary condition provides k , we can assume $k \neq 0$, thus

$$\begin{aligned} \omega &= 0 \rightarrow (k^2 + l^2)U = \beta, \\ l^2 &= \frac{\beta}{U} - k^2 \end{aligned}$$

The sign of l is determined by the radiation condition $\mathbf{c}_g \cdot \hat{\mathbf{y}} > 0$, i.e. waves are traveling away from the boundary, which implies k and l have the same sign and both are real (k is real to match the boundary condition, and l follows from that). Together with the last equation, this implies the Charney-Drazin condition on U that $0 < U < \beta/k^2$. Using these facts, we can write the group velocity relations

$$\mathbf{c}_g \cdot \hat{\mathbf{x}} = 2U \frac{k^2}{k^2 + l^2}, \quad \mathbf{c}_g \cdot \hat{\mathbf{y}} = \frac{2Ukl}{k^2 + l^2}, \quad \frac{\mathbf{c}_g \cdot \hat{\mathbf{y}}}{\mathbf{c}_g \cdot \hat{\mathbf{x}}} = \frac{dy}{dx} = \sqrt{\frac{\beta}{Uk^2} - 1}$$

If we assume generic plane waves $\cos(kx \pm ly)$ and $\sin(kx \pm ly)$, then match the conditions above we find

$$\begin{aligned} q'_x &= \frac{\beta ak}{H_0} \sin(kx), \\ q' &= \frac{-\beta a}{H_0} \cos(kx + ly), \\ v' &= -Uak \sin(kx + ly), \\ u' &= Ual \sin(kx + ly), \text{ as } \nabla \cdot \mathbf{u}' = 0, \\ \nabla \times \mathbf{u}' &= v'_x - u'_y = -Ua \cos(kx + ly)[k^2 + l^2], \\ h' &= -\frac{H_0^2}{f} q' + \frac{H_0}{f} \nabla \times \mathbf{u}' = \frac{H_0}{f} (\beta - U[k^2 + l^2])a \cos(kx + ly) = 0. \end{aligned}$$

Therefore, at least to leading order, these waves have no signature in h' and they are a pure vorticity anomaly.

2 RSWE Waves

Begin with the rotating shallow water equations.

$$\frac{D\mathbf{u}}{Dt} + \mathbf{f} \times \mathbf{u} + g\nabla\eta = 0, \tag{1}$$

$$\frac{Dh}{Dt} + h\nabla \cdot \mathbf{u} = 0. \tag{2}$$

Linearize about zero background flow, assuming a flat bottom with constant Coriolis parameter $\mathbf{f} = f_0 \hat{\mathbf{z}}$. Derive the plane wave dispersion relation.

$$\omega^2 = f^2 + c^2 \kappa^2. \tag{3}$$

Section 2.1.4 may be a helpful guide.

The linearized equations over a flat bottom are

$$\begin{aligned}\frac{\partial \mathbf{u}'}{\partial t} + \mathbf{f} \times \mathbf{u}' + g \nabla h' &= 0, \\ \frac{\partial h'}{\partial t} + H \nabla \cdot \mathbf{u}' &= 0.\end{aligned}$$

Or, more simply

$$\begin{bmatrix} \partial_t & -f & g\partial_x \\ f & \partial_t & g\partial_y \\ H\partial_x & H\partial_y & \partial_t \end{bmatrix} \begin{bmatrix} u' \\ v' \\ h' \end{bmatrix} = 0.$$

If we assume a plane wave $e^{i(kx+ly-\omega t)}$, then

$$\begin{bmatrix} -i\omega & -f & gik \\ f & -i\omega & gil \\ Hik & Hil & -i\omega \end{bmatrix} \begin{bmatrix} u' \\ v' \\ h' \end{bmatrix} = 0.$$

In order to have a nontrivial solution, the determinant must vanish, or

$$-i\omega(-\omega^2 + f^2 + c^2\kappa^2) = 0$$

Which reduces to the dispersion relation as long as $\omega \neq 0$.

3 QG Waves

Begin with the quasigeostrophic equations on the β -plane

$$\frac{Dq}{Dt} = 0, \tag{4}$$

$$q = \nabla^2 \psi - \kappa_D^2 \psi + \beta y, \tag{5}$$

$$u = -\psi_y, \quad v = \psi_x, \tag{6}$$

$$\psi = \frac{gh}{f_0}, \quad \kappa_D = \frac{f}{\sqrt{gH}} \tag{7}$$

Linearize about zero background flow, assuming a flat bottom. Derive the plane Rossby wave dispersion relation:

$$\omega = \frac{-\beta k}{\kappa^2 + \kappa_D^2} \tag{8}$$

The only equation requiring linearization is

$$\begin{aligned}\frac{\partial q'}{\partial t} + \beta v' &= 0, \\ \frac{\partial(\nabla^2 \psi' - \kappa_D^2 \psi')}{\partial t} + \beta \psi'_x &= 0\end{aligned}$$

If we assume a plane wave $e^{i(kx+ly-\omega t)}$, then

$$-i\omega(-[k^2 + l^2]\psi' - \kappa_D^2 \psi') + \beta ik\psi' = 0$$

Thus, if $\psi' \neq 0$,

$$\omega = \frac{-\beta k}{\kappa^2 + \kappa_D^2}.$$

4 Plotting

Plot the dispersion relations from the RSWE and QG solutions above on the same figure. Calculate (numerically) the frequency, phase, and group speeds of RSWE waves (assuming $gH = 10000m^2/s^2$, $f = 10^{-4}/s$) and QG Rossby waves (assuming $f = 10^{-4}/s$, $\beta = 2 \cdot 10^{-11}/sm$, $\kappa_D = 1/100km$) in the following cases:¹

1. $k = 2\kappa_D, l = 0$. (This is approximately a 314 km wavelength).
2. $k = 0.1/m, l = 0$. (This is approximately a 62 m wavelength—typical size of gravity waves, although it technically violates the shallow water approximation).
3. $k = \kappa_D/10, l = 0$ (This is approximately 1 wavelength/Atlantic span).

How long does it take for a wave packet of each type to cross the Atlantic (6000 km) zonally (i.e., in the k direction)?

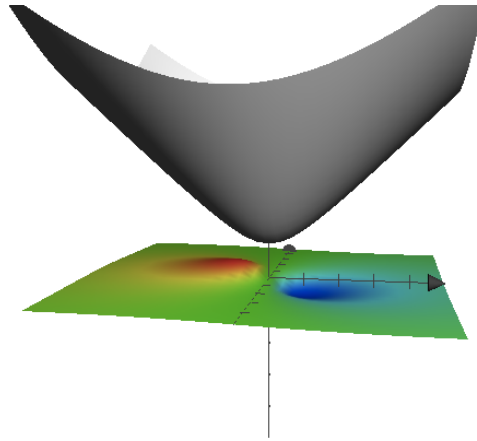


Figure 1: Dispersion relation of Rossby and rotating gravity waves (note—schematic and not to scale!).

1. $k = 2\kappa_D, l = 0$. Rossby: $\omega_R = -8 \cdot 10^{-7}/s = -25.2/yr$, $c_p = 2m/s$, $c_g = 0.024m/s$, $6,000,000m/c_g = 2.5 \cdot 10^8s = 7.93yr$. Gravity: $\omega_G = 0.002/s = 7.2/hr$, $c_p = 100.125m/s$, $c_g = 99.8752m/s$, $6,000,000m/c_g = 60,000s = 16.7hr$.
2. $k = 0.1/m, l = 0$. Rossby: $\omega_R = -2 \cdot 10^{-10}/s = -0.0063/yr$, $c_p = 210^{-9}m/s$, $c_g = 210^{-9}m/s$, $6,000,000m/c_g = 3 \cdot 10^{15}s = 9.510^7yr$, or twice the age of the Earth. Gravity: $\omega_G = 10/s$, $c_p = 100.m/s$, $c_g = 100m/s$, $6,000,000m/c_g = 60,000s = 16.7hr$.
3. $k = \kappa_D/10, l = 0$. Rossby: $\omega_R = -2 \cdot 10^{-7}/s = -6.3/yr$, $c_p = 0.198m/s$, $c_g = 0.194m/s$, $6,000,000m/c_g = 3 \cdot 10^7s = 0.98yr$. Gravity: $\omega_G = 0.00014/s$, $c_p = 141.m/s$, $c_g = 70.7m/s$, $6,000,000m/c_g = 84,852s = 23.5hr$.

¹Note: this value of the deformation wavenumber is much smaller than you'd get for the real ocean. It is similar to that in the real, stratified ocean, or it can be estimated using the *reduced gravity* approximation. See chapter 12 of Cushman-Roisin & Beckers (2010) if you are interested in the details.

5 Zonal Flow

Reconsider the RSWE dispersion relation and the QG dispersion relation by linearizing about a mean zonal flow U instead of a zero background flow. How do the dispersion relations change? (Hint: It is very useful to consider the intrinsic and absolute frequencies in this derivation, see (4.90)).

In the RSWE equations, each prognostic variable is advected by the mean flow. Thus, we can simply follow (4.90) and write

$$\begin{aligned}\hat{\omega} &= \omega - \mathbf{U} \cdot \mathbf{k} = \pm \sqrt{f^2 + c^2 \kappa^2}, \\ \omega &= \mathbf{U} \cdot \mathbf{k} \pm \sqrt{f^2 + c^2 \kappa^2}\end{aligned}$$

Or, we could recognize this as the determinant= 0 of

$$\begin{bmatrix} -i\omega + i\mathbf{U} \cdot \mathbf{k} & -f & gik \\ f & -i\omega + i\mathbf{U} \cdot \mathbf{k} & gil \\ Hik & Hil & -i\omega + i\mathbf{U} \cdot \mathbf{k} \end{bmatrix} \begin{bmatrix} u' \\ v' \\ h' \end{bmatrix} = 0.$$

We note that ω is the absolute frequency, which results from ∂_t , but the cluster $\hat{\omega} = \omega - \mathbf{U} \cdot \mathbf{k}$ appears everywhere in this matrix that linearized about rest matrix equation had ω .

In the QG equations, both the advection of q' and βy are altered potentially,

$$\begin{aligned}\frac{\partial q'}{\partial t} + \mathbf{U} \cdot \nabla q' + \beta(v' + \mathbf{U} \cdot \hat{\mathbf{y}}) &= 0, \\ \frac{\partial(\nabla^2 \psi' - \kappa_D^2 \psi')}{\partial t} + \mathbf{U} \cdot \nabla(\nabla^2 \psi' - \kappa_D^2 \psi') + \beta(\psi'_x + \mathbf{U} \cdot \hat{\mathbf{y}}) &= 0\end{aligned}$$

If \mathbf{U} is zonal, then $\mathbf{U} \cdot \hat{\mathbf{y}} = 0$ and

$$\omega = \frac{-\beta k}{\kappa^2 + \kappa_D^2} + |\mathbf{U}|k.$$

If $\mathbf{U} \cdot \hat{\mathbf{y}} \neq 0$, then things get trickier since the wave equation is no longer homogeneous! How is that possible? It's not, if \mathbf{U} is a solution to the PV equation. That is, to have waves make sense, you must linearize about a *solution* to the equations of motion. $\mathbf{U} \cdot \beta \hat{\mathbf{y}} = 0$ is not a solution to these equations without other effects coming into play (e.g., a background H and thus Q variation). If you add in a background state to cancel the β effect on PV, then you return to a dispersion relation like

$$\omega = \frac{-\beta k}{\kappa^2 + \kappa_D^2} + \mathbf{U} \cdot \mathbf{k}.$$

References

- BÜHLER, OLIVER 2014 *Waves and mean flows*, 2nd edn. *Cambridge monographs on mechanics*. Cambridge, United Kingdom: Cambridge University Press.
- CUSHMAN-ROISIN, BENOIT & BECKERS, J M 2010 *Introduction to geophysical fluid dynamics: Physical and Numerical Aspects*. Academic Press.