Spring 2020 GEOL1820
Homework 1, due Friday, February 7, 9AM

1 Vallis (2019) Problem 1.1

1.1 Show that the derivative of an integral is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1(t)}^{x_2(t)} \varphi(x,t)\,\mathrm{d}x = \int_{x_1}^{x_2} \frac{\partial \varphi}{\partial t}\,\mathrm{d}x + \frac{\mathrm{d}x_2}{\mathrm{d}t}\varphi(x_2,t) - \frac{\mathrm{d}x_1}{\mathrm{d}t}\varphi(x_1,t). \tag{P1.1}$$

By generalizing to three dimensions show that the material derivative of an integral of a fluid property is given by

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V \varphi(\boldsymbol{x},t)\,\mathrm{d}V = \int_V \frac{\partial \varphi}{\partial t}\,\mathrm{d}V + \int_S \varphi \boldsymbol{v}\cdot \mathrm{d}\boldsymbol{S} = \int_V \left[\frac{\partial \varphi}{\partial t} + \nabla\cdot(\boldsymbol{v}\varphi)\right]\mathrm{d}V, \tag{P1.2}$$

where the surface integral ($\int_S$) is over the surface bounding the volume V. Hence deduce that

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V \rho\varphi\,\mathrm{d}V = \int_V \rho \frac{\mathrm{D}\varphi}{\mathrm{D}t}\,\mathrm{d}V. \tag{P1.3}$$

You may assume Leibniz Integral Rule to begin (see Wikipedia for the rule and a proof).

The Leibniz rule is:

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)\,dt\right) = f\big(x,b(x)\big)\cdot\frac{d}{dx}b(x) - f\big(x,a(x)\big)\cdot\frac{d}{dx}a(x) + \int_{a(x)}^{b(x)}\frac{\partial}{\partial x}f(x,t)\,dt,$$

So, when applying it to the problem posed by Vallis, we just swap variable names.

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)\,\mathrm{d}t = \int_{a(x)}^{b(x)}\frac{\partial}{\partial x}f(x,t)\,\mathrm{d}t + f(x,t)\frac{db(x)}{dx} - f(x,t)\frac{da(x)}{dx}$$
$$\frac{d}{dt}\int_{x_1(t)}^{x_2(t)} \varphi(x,t)\,\mathrm{d}x = \int_{x_1(t)}^{x_2(t)}\frac{\partial}{\partial t}\varphi(x,t)\,\mathrm{d}t + \varphi(x,t)\frac{dx_2(t)}{dt} - \varphi(x,t)\frac{dx_1(t)}{dt}$$

In 1D, if the bounds of the integral are taken to follow the *material*, the the rate of change of the bounds is just the fluid velocity.

$$\frac{D}{Dt}\int_{x_1(t)}^{x_2(t)} \varphi(x,t)\,\mathrm{d}x = \int_{x_1(t)}^{x_2(t)}\frac{\partial}{\partial t}\varphi(x,t)\,\mathrm{d}t + \varphi(x,t)u(x_2,t) - \varphi(x,t)u(x_1,t)$$

If we consider a small rectangular solid, then we just need to evaluate the normal velocity at each boundary of the rectangular solid and integrate over each face, noting that for a small enough solid the velocity will be constant over each face or

$$
\begin{aligned}
\frac{D}{Dt}\int_{x_1(t)}^{x_2(t)}\int_{y_1(t)}^{y_2(t)}\int_{z_1(t)}^{z_2(t)}\varphi(\mathbf{x},t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_V \frac{\partial}{\partial t}\varphi(\mathbf{x},t)\,\mathrm{d}V& \\
+\int_{y_1(t)}^{y_2(t)}\int_{z_1(t)}^{z_2(t)}\varphi(\mathbf{x},t)u(x_2,y,z,t)\,\mathrm{d}y\,\mathrm{d}z &- \int_{y_1(t)}^{y_2(t)}\int_{z_1(t)}^{z_2(t)}\varphi(\mathbf{x},t)u(x_1,y,z,t)\,\mathrm{d}y\,\mathrm{d}z \\
+\int_{x_1(t)}^{x_2(t)}\int_{z_1(t)}^{z_2(t)}\varphi(\mathbf{x},t)v(x,y_2,z,t)\,\mathrm{d}x\,\mathrm{d}z &- \int_{x_1(t)}^{x_2(t)}\int_{z_1(t)}^{z_2(t)}\varphi(\mathbf{x},t)v(x,y_1,z,t)\,\mathrm{d}x\,\mathrm{d}z \\
+\int_{x_1(t)}^{x_2(t)}\int_{y_1(t)}^{y_2(t)}\varphi(\mathbf{x},t)w(x,y,z_2,t)\,\mathrm{d}x\,\mathrm{d}y &- \int_{x_1(t)}^{x_2(t)}\int_{y_1(t)}^{y_2(t)}\varphi(\mathbf{x},t)w(x,y,z_1,t)\,\mathrm{d}x\,\mathrm{d}y \\
= \int_V \frac{\partial}{\partial t}\varphi(\mathbf{x},t)\,\mathrm{d}V + \int_S \varphi(\mathbf{x},t)\mathbf{v}\cdot\mathrm{d}\mathbf{S}& \\
= \int_V \left[\frac{\partial}{\partial t}\varphi(\mathbf{x},t) + \nabla\cdot\big(\varphi(\mathbf{x},t)\mathbf{v}\big)\right]\mathrm{d}V&
\end{aligned}
$$

For a more general solid, you have to be very careful about the inner integrals depending on the variables being integrated over in the outer integrals. However, you can just sum up squares like this to any arbitrary shape (Riemann Solids). Finally, using the conservation of mass, and setting $\varphi = \rho\phi$ in the relationship above, we note

$$
\frac{\partial}{\partial t}\varphi + \nabla\cdot(\varphi\mathbf{v}) = \frac{\partial\rho\phi}{\partial t} + \nabla\cdot(\rho\phi\mathbf{v}) = \phi\left[\cancelto{0}{\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})}\right] + \rho\frac{\partial\phi}{\partial t} + \rho\mathbf{v}\cdot\nabla\phi = \rho\frac{D\phi}{Dt}.
$$

Thus,

$$
\frac{D}{Dt}\int_V \rho\phi\,\mathrm{d}V = \int_V \rho\frac{D\phi}{Dt}\,\mathrm{d}V
$$

2 Vallis (2019) Problem 1.2

1.2 *(a)* If molecules move quasi-randomly, why is there no diffusion term in the mass continuity equation?

(b) Suppose that a fluid contains a binary mixture of dry air and water vapour. Show that the change in mass of a parcel of air due to the diffusion of water vapour is exactly balanced by the diffusion of dry air in the opposite direction.

a) Because the velocity is *defined* as tracking the mass of each parcel. If mass could diffuse, then the velocity would not be the velocity. b) In a mixture of two species, then they may diffuse through each other. If ρ_a is the density of just the dry air, and ρ_w is the water vapor density, then

$$\frac{D\rho_a}{Dt} + \rho_a \nabla \cdot \mathbf{v} = \kappa \nabla^2 \rho_a,$$
$$\frac{D\rho_w}{Dt} + \rho_w \nabla \cdot \mathbf{v} = \kappa \nabla^2 \rho_w,$$
$$\rho = \rho_a + \rho_w,$$
$$0 = \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = \frac{D\rho_a}{Dt} + \frac{D\rho_w}{Dt} + \rho_a \nabla \cdot \mathbf{v} + \rho_w \nabla \cdot \mathbf{v}.$$
$$\kappa \nabla^2 \rho_a = \frac{D\rho_a}{Dt} + \rho_a \nabla \cdot \mathbf{v} = -\left(\frac{D\rho_w}{Dt} + \rho_w \nabla \cdot \mathbf{v}\right) = -\kappa \nabla^2 \rho_w$$

Note that the last step does not require any special form of the diffusivity operator, any linear operator acting on each component's density would work similarly, e.g. additional advection by a "slip velocity" $\mathbf{v}_s \cdot \nabla \rho_a$ which would then be equal and opposite for the two constituents.

Hint: the diffusion equation for either of the two constituents is

$$\frac{D\rho_i}{Dt} + \rho_i \nabla \cdot \mathbf{v} = \kappa \nabla^2 \rho_i, \tag{1}$$

and their combined density is $\rho = \rho_a + \rho_w$.

3 Vallis (2019) Problem 1.5

1.5 Show that viscosity will dissipate kinetic energy in a compressible fluid.

Hint: Use integration by parts over the whole fluid volume with no-slip (i.e., velocity is zero on the boundaries), following Section 1.7.3. The correct compressible viscous Navier-Stokes momentum equation is

$$\rho \frac{D\mathbf{v}}{Dt} + \nabla p - \rho g = \mu \nabla^2 \mathbf{v} + \frac{1}{3} \mu \nabla (\nabla \cdot \mathbf{v})$$

Vallis already treats the left side of this equation in (1.73), so let's focus on the right-hand side. We will only prove that energy is dissipated over the whole volume, not pointwise.

$$\int_V \mathbf{v} \cdot \left[\mu \nabla^2 \mathbf{v} + \frac{1}{3} \mu \nabla (\nabla \cdot \mathbf{v})\right] \mathrm{d}V,$$
$$\mu \int_V \mathbf{v} \cdot \left[\nabla \cdot \nabla \mathbf{v} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{v})\right] \mathrm{d}V$$

This is considerably easier if index notation is used with Einstein summation implied, then

$$\mu \int_V \left[v_i \nabla_j \nabla_j v_i + \frac{1}{3} v_i \nabla_i (\nabla_j v_j)\right] \mathrm{d}V,$$

Now we note two different flavors of integration by parts

$$\int_S \left(v_i \nabla_j v_i\right) n_j \,\mathrm{d}S = \int_V \nabla_j \left(v_i \nabla_j v_i\right) \mathrm{d}V = \int_V \left(v_i \nabla_j \nabla_j v_i\right) \mathrm{d}V + \int_V \left(\nabla_j v_i\right)\left(\nabla_j v_i\right) \mathrm{d}V$$
$$\int_S \left(v_i \nabla_j v_j\right) n_i \,\mathrm{d}S = \int_V \nabla_i \left(v_i \nabla_j v_j\right) \mathrm{d}V = \int_V \left(v_i \nabla_i \nabla_j v_j\right) \mathrm{d}V + \int_V \left(\nabla_i v_i\right)\left(\nabla_j v_j\right) \mathrm{d}V$$

and

$$\int_S \left(v_i \nabla_j v_i\right) n_j \,\mathrm{d}S = 0, \qquad \int_S \left(v_i \nabla_j v_j\right) n_i \,\mathrm{d}S = 0$$

Because either the velocity (no-slip) or its normal derivative (slip) vanishes at the boundary. Thus,

$$\int_V \mathbf{v} \cdot \left[\mu \nabla^2 \mathbf{v} + \frac{1}{3}\mu \nabla(\nabla \cdot \mathbf{v})\right] \mathrm{d}V = -\mu \int_V \left(\nabla_j v_i\right)\left(\nabla_j v_i\right) \mathrm{d}V - \frac{\mu}{3} \int_V \left(\nabla \cdot \mathbf{v}\right)\left(\nabla \cdot \mathbf{v}\right) \mathrm{d}V$$

Both of the terms on the RHS are negative definite, so the energy is dissipated. Note that because we used the boundary conditions of no-slip or slip, this proof does not work for an arbitrary volume or for manipulations of the differential equations, so it is possible that these viscous terms could generate energy locally but globally they must dissipate energy.

4 Vallis (2019) Problem 2.2

2.2 *(a)* Show that on Earth we might normally expect the centrifugal term to be much larger than the Coriolis term. Show that if the centrifugal term is incorporated into gravity, and if Earth is a perfect sphere, then gravity is no longer in the local vertical. Estimate the angle by which the apparent gravity differs from the vertical.

(b) If Earth were a perfect sphere, but with otherwise the same distribution of continents and ocean basins, would the distribution of sea level be more or less the same as it is today or would it be radically different?

On Earth $|\Omega| \approx \frac{2\pi}{24hr}$ (approximately because the sidereal day is about 4 minutes less than the solar day which is 24 hours).

a) Note that the local "vertical" Vallis uses here is not the "plumb line" definition of vertical which is the direction a motionless pendulum points, but the direction perpendicular to the surface.

a) The centrifugal force is $\Omega \times \Omega \times \mathbf{r}$ scales as $\Omega^2 r_\perp$ where $r_\perp$ is the minimum radius from the rotation axis. The Coriolis force is $2\Omega \times \mathbf{v}$, which scales as $2\Omega V$. Thus, their ratio is $2V/(\Omega r_\perp)$, which is like a Rossby number. $\Omega r_\perp/2$ ranges from 0 at the pole to $\frac{\pi}{24hr} 6379km \approx 231m/s$. Since most atmospheric and oceanic motions are much slower that $231m/s$, the centrifugal force should be larger.
a2) Estimate the angle. At 30N, $r_\perp$ is $r_e \cos(\theta) = r_e\sqrt(3)/2$. The Coriolis force is thus $\Omega^2 r_\perp (6 \cdot 10^{-5})^2 6 \cdot 10^6 \sqrt{3}/2 m/s^2 \approx 0.02 m/s^2$. Thus, the two vectors to be compared are $-g\hat{\mathbf{r}} + \Omega^2 r_\perp \hat{\mathbf{r}}_\perp$ and $-g\hat{\mathbf{r}}$. The triangle with its base aligned with $\hat{\mathbf{r}}_\perp$ and its hypotenuse aligned along $\hat{\mathbf{r}}$, which are

separated by an angle of 30 degrees is what we want to think about. The component of g along the base is $-g\cos(30) = -g\sqrt{3}/2$ and along the other side is $-g\sin(30) = -g/2$. When we include the Coriolis force, we think about the smaller right triangle with sides $-g/2$ and $-g\sqrt{3}/2 + 0.02$, so the new angle is $\tan^{-1}[-g/2/(-g\sqrt{3}/2 + 0.02m/s^2)] = \tan^{-1}[1/(\sqrt{3} - 0.004)] = 30.06°$. So, the deviation in this case is about 0.002 degrees or 0.00003 radians.
b) If Earth were a perfect sphere, the oceans would sense that the equator was "downhill", i.e., at a lower potential because of the centrifugal force. After readjusting, the oceans at the poles would be shallow and those at the equator would be very deep. Instead, on the real earth, the solid earth is "level" when it is parallel to the combined gravitational and centrifugal potentials (i.e., isostasy), which makes the ocean roughly the same depth at the poles and at the equator.

5 Vallis (2019) Problem 2.7

> 2.7 Show that the inviscid, adiabatic, hydrostatic primitive equations for a compressible fluid conserve a form of energy (kinetic plus potential plus internal), and that the kinetic energy has no contribution from the vertical velocity. Obtain an explicit form for the conserved energy, and provide a physical interpretation for this result. (You may assume Cartesian geometry and a uniform gravitational field in the vertical direction.)

> Alternatively, use the hydrostatic Boussinesq equations and again show that the vertical velocity does not contribute. Obtain an explicit form for the conserved energy and interpret your result.

Hint: follow the procedure from Section 1.7, but beginning with equations 2.41. You can use the tangent plane form, rather than the spherical (i.e., 2.43a and 2.43b instead of 2.41a and 2.41b). Also, you can use either the compressible form or the Boussinesq form–the latter will be more directly applicable in this class.

Compressible:

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$
$$\frac{Dv}{Dt} - fu = -\frac{1}{\rho}\frac{\partial p}{\partial y},$$
$$\rho\frac{\partial \Phi}{\partial z} = \frac{\partial p}{\partial z},$$

We form a horizontal kinetic energy equation by multiplying the first equation by u, and the second by v, and we recall (1.69) and (1.70)

$$
\begin{aligned}
\frac{\rho}{2}\frac{D\mathbf{u}_h^2}{Dt} &= -\mathbf{u}_h\cdot\nabla p,\\
\rho\frac{DI}{Dt} &= -p\nabla\cdot\mathbf{v} = -p\nabla\cdot\mathbf{u}_h - p\frac{\partial w}{\partial z},\\
\rho\frac{D\Phi}{Dt} &= -\rho\mathbf{v}\cdot\nabla\Phi = -w\frac{\partial p}{\partial z}.
\end{aligned}
$$

Thus,

$$\rho\frac{D}{Dt}\left(\frac{u_h^2}{2}+I+\Phi\right) = -\nabla\cdot(p\mathbf{v}),$$

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{u_h^2}{2}+I+\Phi\right)\right]+\nabla\cdot\left[\mathbf{v}\left(\rho\left(\frac{u_h^2}{2}+I+\Phi\right)+p\right)\right]=0$$

Which is the flux form of the hydrostatic energy.

Boussinesq:

$$
\begin{aligned}
\frac{Du}{Dt} - fv &= -\frac{\partial\phi}{\partial x},\\
\frac{Dv}{Dt} - fu &= -\frac{\partial\phi}{\partial y},\\
\frac{\partial\phi}{\partial z} &= b,\\
\frac{Db}{Dt} &= 0,\\
\nabla\cdot\mathbf{v} &= 0.
\end{aligned}
$$

We form a horizontal kinetic energy equation by multiplying the first equation by u, and the second by v, and we recall (1.69) and (1.70)

$$
\begin{aligned}
\frac{1}{2}\frac{D\mathbf{u}_h^2}{Dt} &= -\mathbf{u}_h\cdot\nabla\phi,\\
0 &= wb - w\frac{\partial\phi}{\partial z},\\
\frac{D(-z)}{Dt} &= -w,\\
\frac{D(-zb)}{Dt} &= -wb,
\end{aligned}
$$

Thus,

$$\frac{\partial}{\partial t}\left(\frac{u_h^2}{2}-zb\right)+\nabla\cdot\left[\mathbf{v}\left(\frac{u_h^2}{2}-zb+\phi\right)\right]=0$$

So, the Boussinesq system has kinetic energy (which in the hydrostatic case is only from horizontal velocity), and potential energy ($-zb$), which are transported by the 3D velocity field. Internal energy does not appear, as its conversion vanishes due to $\nabla\cdot\mathbf{v}=0$.

6 Vallis (2019) Problem 2.8

2.8 *(a)* Consider a scalar field, like temperature, T. Explain in words why the material derivative in a rotating frame is equal to the material derivative in the inertial frame; that is, explain why $(DT/Dt)_I = (DT/Dt)_R$.

(b) The material derivative of a scalar is given by $\partial T/\partial t + (\boldsymbol{v} \cdot \nabla)T$. Show (mathematically, with equations) that the individual terms are different in the rotating and inertial frames (and obtain an expression for how much) but that their sum is the same.

b) Hint: write out the velocity in each frame explicitly using (2.7): $\mathbf{v}_I = \mathbf{v}_R + \boldsymbol{\Omega} \times \mathbf{r}$

a) The value of a scalar is unaffected by what direction it is viewed from. Thus, a rotating observer and a fixed observer see the same value. The material derivative of a scalar describes the rate of change of the scalar *following the motion of the fluid*. Given that both observers agree about the motion of the fluid (although they disagree about how to write the components of velocity), the change (material derivative) of the scalar is likewise unaffected by rotation.
b) Here we use $\mathbf{v}_I = \mathbf{v}_R + \boldsymbol{\Omega} \times \mathbf{r}$

$$
\begin{aligned}
\frac{DT}{Dt} =& \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \\
& \frac{\partial T}{\partial t}_I + \mathbf{v}_I \cdot \nabla_I T = \frac{\partial T}{\partial t}_I + (\mathbf{v}_R + \Omega \times \mathbf{r}) \cdot \nabla T \\
& \frac{\partial T}{\partial t}_I = \frac{\partial T}{\partial t}_R + \Omega \times \mathbf{r} \cdot \nabla T
\end{aligned}
$$

The last relationship is seen by using the chain rule on the time derivative of the tracer. If the tracer is stationary in the inertial frame, then its location is moving in the rotating frame, and vice versa. So, the partial derivative with respect to time, while holding position fixed, means something different in each frame. Thus, you need to account for the relative motion of the coordinate systems, i.e., a partial derivative also differentiates the moving coordinate location,

$$
\frac{\partial T(\mathbf{x}_R(t), t)}{\partial t} = \frac{\partial \mathbf{x}_R}{\partial t} \cdot \nabla T + \frac{\partial \mathbf{T}}{\partial t} = \Omega \times \mathbf{r} \cdot \nabla T + \frac{\partial \mathbf{T}}{\partial t}
$$

7 Vallis (2019) Problem 3.1

3.1 Consider two-dimensional, incompressible, fluid flow in a rotating frame of reference on the f-plane. Linearize the equations about a state of rest to obtain the momentum equations:

$$\frac{\partial u}{\partial t} - fv = -\frac{\partial \phi}{\partial x}, \qquad \frac{\partial v}{\partial t} + fu = -\frac{\partial \phi}{\partial y}. \tag{P3.1}$$

(a) Ignore the pressure term and determine the solution to the resulting equations. Show that the speed of fluid parcels is constant. Show that the trajectory of the fluid parcels is a circle with radius $|U|/f$, where $|U|$ is the fluid speed. (These solutions are inertial oscillations.)
(b) What is the period of oscillation of a fluid parcel?
(c) If parcels travel in straight lines in inertial frames, why is the answer to (b) not equal to the rotation period of the frame of reference? (See also Problem 4.3.)

a) Hint: Without loss of generality, align the coordinate system so that the initial velocity is in the x direction: $u_0 = |U|, v_0 = 0$.

First linearize:

$$\frac{D\mathbf{v}}{Dt} + \mathbf{f} \times \mathbf{v} = -\nabla\phi \approx \frac{\partial \mathbf{v}}{\partial t} + \mathbf{f} \times \mathbf{v}$$

a) Solve the following assuming $u(t=0) = U, v(t=0) = 0, \phi = 0$. Noting that U is independent of space, and there are no spatial derivatives remaining, the system is now just the ODEs:

$$\begin{aligned}
\frac{du}{dt} - fv &= 0, & \frac{dv}{dt} + fu &= 0,\\
\frac{d^2u}{dt^2} - f\frac{dv}{dt} = \frac{d^2u}{dt^2} + f^2u &= 0, & \frac{d^2v}{dt^2} + f\frac{du}{dt} = \frac{d^2v}{dt^2} + f^2v &= 0,\\
u &= U\cos(ft), & v &= -U\sin(ft)\\
\Delta x = \int u\,\mathrm{dt} &= U/f\sin(ft), & \Delta y = \int v\,\mathrm{dt} &= U/f\cos(ft)
\end{aligned}$$

These trajectories inscribe a circle.
b) f is the angular frequency, so $2\pi/f$ is the period (12 hours at the pole, i.e., twice per day).
c) Parcels travel in straight lines in the inertial frame, but the Coriolis force alone is not sufficient to reflect the straight lines, the combination of Coriolis and centrifugal are required. Combining both of those provides spiral patterns whose period is the rotation period.

8 Vallis (2019) Problem 3.3

3.3 Consider a rapidly rotating (i.e., in near geostrophic balance) Boussinesq fluid on the f-plane.

(a) Show that the pressure divided by the density scales as $\phi \sim fUL$.

(b) Show that the horizontal divergence of the geostrophic wind vanishes. Thus, argue that the scaling $W \sim UH/L$ is an *overestimate* for the magnitude of the vertical velocity. Obtain a scaling estimate for the magnitude of vertical velocity in rapidly rotating flow.

(c) Using these results, or otherwise, discuss whether hydrostatic balance is more or less likely to hold in a rotating flow than in non-rotating flow.

a) If we scale the geostrophic relation:

$$\underbrace{\mathbf{f} \times \mathbf{v}}_{fU} = -\underbrace{\nabla\phi}_{\phi L^{-1}}$$

For these terms to balance, we must have $\phi \sim fUL$.

b) Taking the cross derivatives of the pressure term, we find horizontal incompressibility,

$$-f\frac{dv_g}{dy} = -\frac{d^2\phi}{dxdy}, \qquad f\frac{du_g}{dx} = -\frac{d^2\phi}{dxdy},$$

$$\frac{du_g}{dx} + \frac{dv_g}{dy} = \frac{1}{f}\left[-\frac{d^2\phi}{dxdy} + \frac{d^2\phi}{dxdy}\right] = 0$$

Thus, although each of the terms on the left scales as U/L their sum is zero (or small for finite Rossby number), and so the horizontal terms of the 3D divergence $\nabla_3 \cdot \mathbf{v}$ sum up to less than U/L, so $W/H \ll U/L$.

c) The conditions for hydrostatic balance to hold were that $Dw/Dt \ll -\frac{\partial\phi}{\partial z}$ was less than the pressure gradient. In geostrophic balance, the pressure gradient is large and $\phi \sim fUL$, thus $\frac{\partial\phi}{\partial z} \sim fUL/H$. If $U \ll fL$, then $fUL/H \gg U^2/H$. U^2/H is in turn much greater than WW/H or WU/L if $W/H \ll U/L$. The W scale is $RoHU/L$, as in (3.47), because the ageostrophic horizontal velocity, which can converge, is Rossby smaller than the geostrophic.

9 Vallis (2019) Problem 3.7

3.7 (a) Estimate the magnitude of the zonal thermal wind 5 km above the surface in the midlatitude atmosphere in summer and winter using approximate values for the meridional temperature gradient in Earth's atmosphere.

(b) Repeat the exercise for Venus and Mars.

(c) Repeat the exercise for Earth's ocean, 1 km below the surface.

b) For Mars, you can take $T = -125°C$ at the pole and $T = 20°C$ at the equator (https://www.space.com/16907-what-is-the-temperature-of-mars.html). For Venus, you can use $T = 460°C$ at the pole and $T = 460°C$ at the equator (https://www.universetoday.com/14306/temperature-of-venus/). No need to do both summer & winter.

To address these problems, we can simplify the log-pressure coordinate version of the thermal wind: $f\frac{\partial u_g}{\partial Z} \approx f\frac{u_g}{H} = -\frac{R}{H}\nabla_Z T$, where we assume that $u_g \approx 0$ on the ground. Thus, $u_g \approx \frac{R}{f}\nabla_Z T \approx \frac{R\Delta T}{fr\pi/2} \approx \frac{183m^2/s^2/K\Delta T}{fr}$. a) On earth, $f \approx 1 \cdot 10^{-4}/s$ and $r = 6 \cdot 10^6 m$, so $u_g \approx 0.3m/s/K\Delta T$. Using $18°C$ for the equator and $0°C$ for the summer pole and $-40°C$ for the winter pole, the log-pressure coordinates thermal wind gives $u_g \approx 6 - 20m/s$. This is an underestimate of the westerlies, which occupy less than 1/4 of the circumference of the Earth.
b) On Mars, the day is only 37 minutes longer, so $f \approx 1 \cdot 10^{-4}/s$ is still OK, but $r = 3 \cdot 10^6 m$. Thus, $u_g \approx 0.6m/s/K\Delta T$, which is $90m/s$–much faster than Earth! On Venus, the temperatures are the same so no zonal winds are expected.
c) For the ocean, the temperature differences are about the same as the summertime Earth's atmosphere (20K), however, you can't use the ideal gas law. Instead, you need to calculate the buoyancy difference to use the Boussinesq approximation. Using the equation of state from Vallis equation (1.42), $\Delta b = g\beta_T \Delta T = 0.03m/s^2$. Then $u_g \approx H\Delta b/(fr\pi/2) \approx 1s\Delta b$, so $0.03m/s$.

References

Vallis, G. K. (2019). *Essentials of Atmospheric and Oceanic Dynamics*. Cambridge University Press.