Spring 2020 GEOL1820
Homework 2, due Monday, February 24, 9AM

1 Vallis (2019) Problem 3.2

3.2 Consider a dry, hydrostatic, ideal-gas atmosphere whose lapse rate is one of constant potential temperature. What is its vertical extent? That is, at what height does the density vanish? Is this result a problem for the any of the assumptions we have made in the first three chapters?

The constant potential temperature adiabatic lapse rate is (3.57):

$$0 = \frac{T}{\theta}\frac{\partial \theta}{\partial z} = \frac{\partial T}{\partial z} + \frac{g}{c_p},$$
$$\frac{\partial T}{\partial z} = -\frac{g}{c_p},$$
$$T = T_s - \frac{g}{c_p}z$$

This stratification is *neutrally stable.* The ideal gas law and hydrostatic balance combine to yield

$$p = \rho R T,$$
$$\frac{\partial p}{\partial z} = -\rho g,$$
$$\frac{\partial \ln(p)}{\partial z} = \frac{1}{p}\frac{\partial p}{\partial z} = \frac{1}{RT} = \frac{c_p/R}{c_p T_s/g - z},$$

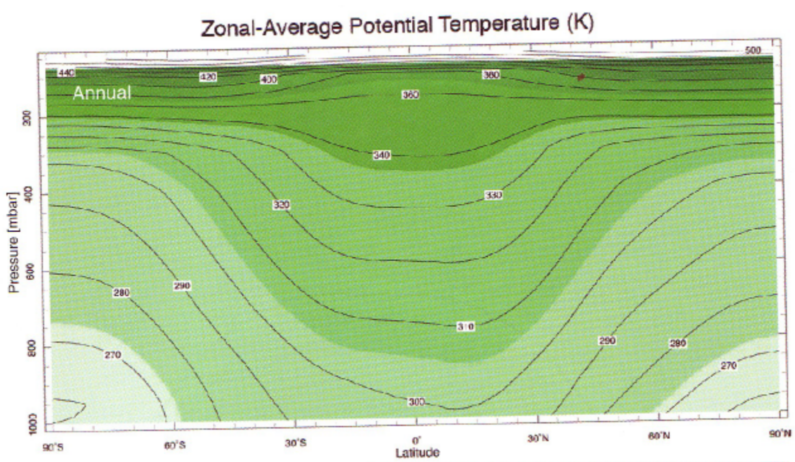
Integrating the last equation in z yields

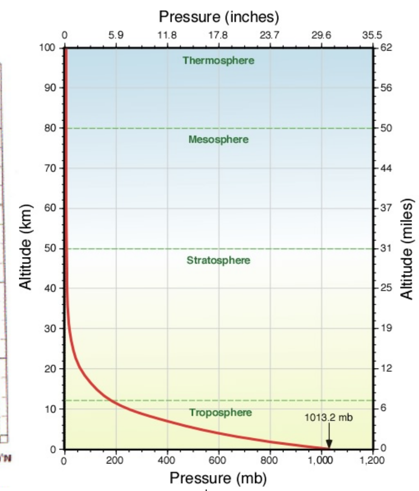
$$\ln\frac{p}{p_s} = \ln\left[\left(\frac{c_p T_s}{g} - z\right)^{-c_p/R}\right],$$
$$\ln\frac{p}{p_s} = \ln\left[(29.5km - z)^{-3.5}\right].$$

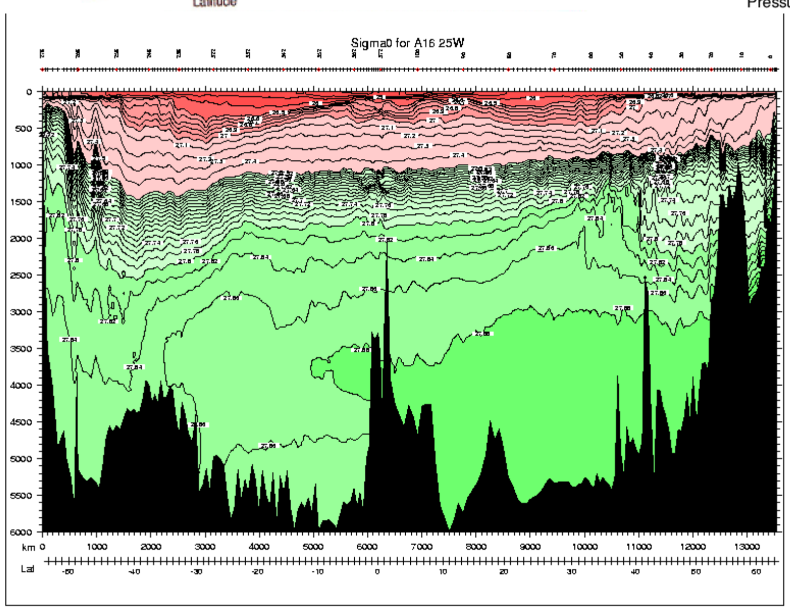
Which means that the pressure decreases to zero near 29.5 km. This result is not a problem for a thin atmosphere approximation, as this height of the atmosphere is small compared to the earth's radius. It is a problem for the Boussinesq approximation if motions are a sizeable fraction of 29.5 km, which would be true of warm anomalies in this system, as the background is neutrally buoyant (i.e., a warm air parcel would continue to rise to the top). The atmospheric boundary layer (which is a strange concept in a neutrally-stable atmosphere, but anyway) might still have the Boussinesq approximation apply. In the real world, the stratosphere–where potential temperature increases due to radiation absorption causing stable stratification–prevents air masses from ascending to the top of the atmosphere. Even the anelastic equations are a bit funny as $\rho \to 0$ is approached, although they are better suited than the Boussinesq equations.

2 Vallis (2019) Problem 3.5

3.5 Using approximate but realistic values for the observed stratification, calculate the buoyancy period for (a) the midlatitude troposphere, (b) the stratosphere, (c) the oceanic thermocline, (d) the oceanic abyss.







A little googling yields this section of atmospheric potential temperature (Marshall & Plumb, 2008) and Atlantic potential density (Talley et al., 2011).

a) Midlatitude Troposphere: 70K warmer over 800 mb of pressure, which is about 10 km high. So, $N^2 = \frac{9.81m/s^2}{300K}\frac{70K}{10km} \approx 2 \cdot 10^{-4}s^{-2}$, $N = 0.015s^{-1} = \frac{2\pi}{400s}$.

b) Stratosphere: 70K warmer over 200 mb of pressure, which is about 30 km high. So, $N^2 = \frac{9.81m/s^2}{400K}\frac{70K}{30km} \approx 6 \cdot 10^{-5}s^{-2}$, $N = 0.007s^{-1} = \frac{2\pi}{800s}$.

c) Ocean Pycnocline: potential density (anomaly) of $26.4kg/m^3$ to 27.4 over 700m of depth. So, $N^2 = \frac{9.81m/s^2}{1027kg/m^3}\frac{1kg/m^3}{700m} \approx 1 \cdot 10^{-5}s^{-2}$, $N = 0.004s^{-1} = \frac{2\pi}{1700s}$

d) Ocean Abyss: potential density (anomaly) of $27.84kg/m^3$ to 27.88 over 3000m of depth (note–really should use a deeper reference pressure potential density, but it's just an estimate). So, $N^2 = \frac{9.81m/s^2}{1028kg/m^3}\frac{0.04kg/m^3}{3000m} \approx 1 \cdot 10^{-7}s^{-2}$, $N = 0.0004s^{-1} = \frac{2\pi}{18{,}000s}$

3 Vallis (2019) Problem 4.4

4.4 Linearize the f-plane shallow water system about a state of rest. Suppose that there is an initial disturbance given in the general form

$$\eta = \iint \tilde{\eta}_{k,l} \mathrm{e}^{\mathrm{i}(kx+ly)} \, \mathrm{d}k \, \mathrm{d}l, \tag{P4.2}$$

where η is the deviation surface height and the Fourier coefficients $\tilde{\eta}_{k,l}$ are given, and that the initial velocity is zero.

(a) Obtain the geopotential field at the completion of geostrophic adjustment, and show that the deformation scale is a natural length scale in the problem.

(b) Show that the change in total energy during the adjustment is always less than or equal to zero. Neglect any initial divergence.
N.B. Because the problem is linear, the Fourier modes do not interact.

$$\frac{D\mathbf{u}}{Dt} + \mathbf{f} \times \mathbf{u} = -g\nabla\eta,$$
$$\frac{Dh}{Dt} + h\nabla \cdot \mathbf{u} = 0,$$
$$\frac{DQ}{Dt} = 0.$$

Linearized over a flat bottom ($h = H + \eta$), these equations become

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{f} \times \mathbf{u} = -g\nabla\eta,$$
$$\frac{\partial \eta}{\partial t} + H\nabla \cdot \mathbf{u} = 0,$$
$$\frac{\partial}{\partial t} q = \frac{\partial}{\partial t}\left(\zeta - f_0 \frac{\eta}{H}\right) = 0$$

The initial conditions are

$$\eta = \iint \tilde{\eta}_{k,j} e^{ikx+ily} \, \mathrm{d}k \, \mathrm{d}l$$

a) Obtain the geopotential field at the end of geostrophic adjustment. Note that the different modes do not interact, so we can just analyze the state of one mode at a time and sum them up at the end. Thus,

$$q_{k.l} = -f_0 \frac{\tilde{\eta}_{k,j}}{H} e^{ikx+ily} = \zeta_{k,l} - f_0 \frac{\eta_{k,l}}{H}$$

Where I use the notation that a k, l subscript without a tilde includes the $e^{ikx+ily}$ factor and is a function of time if needed. For the final state to be in geostrophic balance, we must have

$$\zeta_{k.l} = \frac{\partial v_{k,l}}{\partial x} - \frac{\partial u_{k,l}}{\partial y} = \frac{g}{f_0}\nabla^2 \eta_{k,l} = -\frac{g}{f_0}\left(k^2 + l^2\right)\eta_{k,l}$$

Note, this relies on the waves "propagating away", which doesn't make sense in an integral over an infinite domain. So, we should imagine that there are some limits on the area integrals or that the waves are damped otherwise. Thus, the end state for each mode is:

$$-\frac{f_0}{H}\tilde{\eta}_{k,l}e^{ikx+ily} = -\left[\frac{g}{f_0}(k^2+l^2) + \frac{f_0}{H}\right]\eta_{k,l},$$
$$\eta_{k,l} = \frac{\tilde{\eta}_{k,l}}{1 + L_d^2(k^2+l^2)}e^{ikx+ily},$$
$$\eta = \iint \frac{\tilde{\eta}_{k,l}}{1 + L_d^2(k^2+l^2)}e^{ikx+ily}\,\mathrm{d}k\,\mathrm{d}l.$$

Thus, the final state looks a lot like the initial state, but scaled to decrease to zero as $(k^2+l^2)^{-1}$ becomes greater than L_d^2, i.e., for scales much smaller than the deformation radius. For scales much larger than the deformation radius, the final state is the same as the initial state!

b) Show that the energy decreases. The total energy is the area integral of the kinetic and potential energies (4.26), which is proportional to the amplitude of the Fourier integrals by Parseval's theorem:

$$\frac{d\hat{E}}{dt} = \frac{1}{2}\frac{d}{dt}\iint (h\mathbf{u}^2 + gh^2)\,\mathrm{d}x\,\mathrm{d}y \approx \iint (H\mathbf{u}^2 + g\eta^2)\,\mathrm{d}x\,\mathrm{d}y \propto \iint (H|\mathbf{u}_{k,l}|^2 + g|\eta_{k,l}|^2)\,\mathrm{d}k\,\mathrm{d}l$$

The squared velocities of the final state are in geostrophic balance, thus

$$|u_{k,l}|^2 = \frac{g^2}{f_0^2}\left(\frac{\partial \eta}{\partial y}\right)^2 = \frac{gL_d^2 l^2}{H}|\eta_{k,l}|^2,$$
$$|v_{k,l}|^2 = \frac{g^2}{f_0^2}\left(\frac{\partial \eta}{\partial x}\right)^2 = \frac{gL_d^2 k^2}{H}|\eta_{k,l}|^2.$$

Thus, the initial energy is

$$\frac{1}{2}\iint (g\eta_{k,l}^2)\,\mathrm{d}k\,\mathrm{d}l = \frac{1}{2}\iint (g|\tilde{\eta}_{k,l}|^2\left|e^{ikx+ily}\right|^2)\,\mathrm{d}k\,\mathrm{d}l = \frac{1}{2}\iint (g|\tilde{\eta}_{k,l}|^2\,\mathrm{d}k\,\mathrm{d}l$$

and the final energy is

$$\frac{1}{2}\iint \left[L_d^2(k^2+l^2)+1\right] g|\eta_{k,l}|^2\,\mathrm{d}k\,\mathrm{d}l = \frac{1}{2}\iint \frac{1+L_d^2(k^2+l^2)}{\left(1+L_d^2(k^2+l^2)\right)^2} g|\tilde{\eta}_{k,l}|^2\,\mathrm{d}k\,\mathrm{d}l$$
$$= \frac{1}{2}\iint \frac{1}{1+L_d^2(k^2+l^2)} g|\tilde{\eta}_{k,l}|^2\,\mathrm{d}k\,\mathrm{d}l$$

So, the total energy is conserved when $(k^2+l^2)^{-1} \gg L_d^2$ and all energy is lost when $k^2+l^2 \to \infty$.

4 Vallis (2019) Problem 4.5

4.5 If energy conservation is one of the most basic physical laws, how can energy be lost in geostrophic adjustment?

The energy is not lost, it propagates away. In the preceding problem, this is a bit confusing since the waves were removed by finding the final PV, but the domain was infinite. As mentioned above, the waves need to be damped to arrive at a final geostrophically balanced state. If the waves propagate away to infinity, or are selectively damped without affecting the PV, then the solution above works. In both cases, energy is lost along with the removed waves.

5 Vallis (2019) Problem 4.7

4.7 In the shallow water equations show that, if the flow is approximately geostrophically balanced, the energy at large scales is predominantly potential energy and the energy at small scales is predominantly kinetic energy. Define precisely what 'large scale' and 'small scale' mean in this context.

As shown above,

$$\hat{E} = \frac{1}{2}\iint \left[\underbrace{L_d^2(k^2+l^2)}_{\text{KE}} + \underbrace{1}_{\text{PE}}\right] g|\eta_{k,l}|^2 \,\mathrm{d}k\,\mathrm{d}l$$

Thus "large" scale is $(k^2+l^2)^{-1} \gg L_d^2$ and "small" scale is $(k^2+l^2)^{-1} \ll L_d^2$.

6 Vallis (2019) Problem 4.8

4.8 In the shallow water geostrophic adjustment problem, show that at large scales the velocity essentially adjusts to the height field, and that at small scales the height field essentially adjusts to the velocity field. Your derivation may be detailed and mathematical, but explain the result at the end in words and in physical terms.

Thus, recalling the end state of PV for each mode from above:

$$-\frac{f_0}{H}\tilde{\eta}_{k,l}e^{ikx+ily} = -\left[\underbrace{\frac{g}{f_0}(k^2+l^2)}_{\text{final velocity}} + \underbrace{\frac{f_0}{H}}_{\text{final height}}\right]\eta_{k,l},$$

$$\underbrace{\eta_{k,l}}_{\text{initial}} = \frac{1}{\underbrace{1}_{\text{final height}} + \underbrace{L_d^2(k^2+l^2)}_{\text{final velocity}}}\underbrace{\eta_{k,l}}_{\text{final}}.$$

At large scales in geostrophic balance, the PV and energy are dominated by the height field/potential energy. At small scales in geostrophic balance, the PV and energy are dominated by the velocity contribution and the kinetic energy contribution. So, when considering how the final PV matches the initial PV, it is a function of scale, where the larger scales have PV dominated by height so the velocity adjusts to the height while the smaller scales have PV dominated by velocity so the height adjusts.

7 Cushman-Roisin (1994) Problem 3-1

3-1. A laboratory tank consists of a cylindrical container 30 cm in diameter, filled at rest by 20 cm of fresh water and then spun at 30 rpm. After a state of solid-body rotation is achieved, what is the difference in water level between the rim and the center? How does this difference compare with the minimum depth at the center?

In the final state, we match the centrifugal force to the pressure gradient force (shallow water equation form), both of which are in the radial direction. The centrifugal force is $\Omega^2 r$. The pressure gradient force is (in cylindrical coordinates) $g\nabla\eta = g\frac{d\eta}{dr}$. Thus,

$$\Omega^2 r = g\frac{d\eta}{dr},$$

$$\eta = \frac{\Omega^2}{2g}r^2 = \frac{0.5}{m}(0.3m)^2 = 0.045m = 0.22H$$

So, the outer tank is about 4 centimeters deeper than the center, or about 22% of the depth. To find the depth at the center, one needs to conserve the total volume, which requires the following integral.

$$\iint (H+\eta) r \,\mathrm{d}r\,\mathrm{d}\theta = \iint (Hr + \frac{\Omega^2}{2g}r^3)\,\mathrm{d}r\,\mathrm{d}\theta = 2\pi\int_0^R (Hr + \frac{\Omega^2}{2g}r^3)\,\mathrm{d}r = \pi(HR^2 + \frac{\Omega^2}{4g}R^4)$$

In the beginning the volume is $\pi R^2(0.2m)$, at the end it is the formula above, which means that $H = 0.177$ m, or just a bit more than 2 centimeters (10%) shallower than in the non-rotating tank.

8 Cushman-Roisin (1994) Problem 3-3

3-3. Using the scale given in (3-34), compare the dynamic pressure induced by the Gulf Stream (speed = 1 m/s, width = 40 km, and depth = 500 m) with the main hydrostatic pressure due to the weight of the same water depth. Also, convert the dynamic-pressure scale to its equivalent height of hydrostatic pressure (head). What can you infer about the possibility of measuring oceanic dynamic pressures by a pressure gauge?

The scale in C-R is $P = \rho_0\Omega LU$, which is the same as Vallis (2019) (3.49), yields a pressure difference across the GS of $2900kg/m/s^2 = 2900Pa$. The hydrostatic pressure of the GS is $p = \rho g H = 5{\cdot}10^6 Pa$.

The equivalent height of pressure from the dynamic pressure is $\rho_0 \Omega L U/\rho/g = 0.3m$, which is a difficult sea surface height difference to detect over 50 km of distance. It is, however, possible to do so with 2 digits of precision with modern satellite altimeters.

9 Cushman-Roisin and Beckers (2011) Problem 3-5

3-5. Within the Boussinesq approximation and for negligible diffusion in (3.24), show that for an ocean at rest, density can only be a function of depth: $\rho = \rho(z)$. (*Hint*: The situation at rest is characterized by the absence of movement and temporal variations.)

The equation at hand is $D\rho/Dt = \kappa \nabla^2 \rho$. If the fluid is at rest, then $D\rho/Dt = 0$. If there is a horizontal contribution to $\nabla^2 \rho$, then it will drive a current through the thermal wind relation or through horizontal pressure gradients. While it is possible to cancel out the horizontal density gradient contributions to pressure at one depth, it is not possible to do so at all (hydrostatic relation). Thus, the density ρ obeys $d^2\rho/dz^2 = 0$, which means that it can only be a function of z (indeed, only a simple function for smooth κ!).

References

Cushman-Roisin, B. (1994). *Introduction to geophysical fluid dynamics.* Prentice Hall, Englewood Cliffs, N.J.

Cushman-Roisin, B. and Beckers, J.-M. (2011). *Introduction to geophysical fluid dynamics: physical and numerical aspects*, volume v. 101 of *International geophysics series.* Academic Press, Waltham, MA, 2nd ed edition.

Vallis, G. K. (2019). *Essentials of Atmospheric and Oceanic Dynamics.* Cambridge University Press.