

Spring 2020 GEOL1820
Homework 3, due Friday, March 6, 9AM

1 Vallis (2019) Problem 5.1a

5.1 Do either or both:

- (a) Carry through the derivation of the quasi-geostrophic system starting with the anelastic equations and obtain (5.66).

In each case, state the differences between your results and the Boussinesq result.

We begin with the anelastic equations

$$\begin{aligned} \frac{D\mathbf{u}}{Dt} + \mathbf{f} \times \mathbf{u} &= -\nabla\phi, \\ \frac{\partial\phi}{\partial z} &= b, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{\tilde{\rho}} \frac{\partial\tilde{\rho}w}{\partial z} &= 0, \\ \frac{Db'}{Dt} + wN^2 &= 0 \end{aligned}$$

To leading order, we have

$$\begin{aligned} \mathbf{f}_0 \times \mathbf{u}_g &= -\nabla\phi, \\ \nabla \cdot \mathbf{u}_g &= 0, \\ \frac{\partial\tilde{\rho}w}{\partial z} &= 0. \end{aligned}$$

And thus w is much smaller than the magnitude of \mathbf{u}_g . As in the Boussinesq case, the vertical vorticity equation arrives by differentiating the horizontal momentum equations, which is (5.58)

$$\begin{aligned} \frac{D(\zeta + f)}{Dt} &= -(\zeta + f) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \left(\frac{\partial u}{\partial z} \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \frac{\partial w}{\partial x} \right), \\ \frac{D_g(\zeta_g + f)}{Dt} &= \frac{f_0}{\tilde{\rho}} \frac{\partial\tilde{\rho}w}{\partial z}. \end{aligned}$$

And if we approximate the leading order buoyancy equation,

$$\begin{aligned} \frac{D_g b'}{Dt} + wN^2 &= 0, \\ \frac{D_g(\zeta_g + f)}{Dt} &= -\frac{f_0}{\tilde{\rho}} \frac{\partial}{\partial z} \left(\frac{\tilde{\rho}}{N^2} \frac{D_g b'}{Dt} \right) = -\frac{f_0}{\tilde{\rho}} \frac{\partial}{\partial z} \frac{D_g}{Dt} \left(\frac{\tilde{\rho} b'}{N^2} \right) = -\frac{f_0}{\tilde{\rho}} \frac{D_g}{Dt} \frac{\partial}{\partial z} \left(\frac{\tilde{\rho} b'}{N^2} \right) - \frac{f_0}{\tilde{\rho}} \frac{\partial \mathbf{u}_g}{\partial z} \cdot \nabla \left(\frac{\tilde{\rho} b'}{N^2} \right) \end{aligned}$$

Where the middle steps come in noting that $\tilde{\rho}$ and N^2 depend only on z , not time or horizontal directions (by definition or to leading order, either can be justified). Note that the thermal wind relationship makes the last term vanish, as

$$-\frac{f_0}{\tilde{\rho}} \frac{\partial \mathbf{u}_g}{\partial z} \cdot \nabla \left(\frac{\tilde{\rho} b'}{N^2} \right) = -\frac{f_0}{N^2} \frac{\partial \mathbf{u}_g}{\partial z} \cdot (\nabla b') = \frac{f_0}{N^2} \frac{\partial \mathbf{u}_g}{\partial z} \cdot \left(\mathbf{f}_0 \times \frac{\partial \mathbf{u}_g}{\partial z} \right) = 0$$

Thus, combining the left and right sides of the equation before last we arrive at the anelastic QGPV equation,

$$\frac{D_g q}{Dt} = 0, \quad q = \zeta_g + f + \frac{f_0^2}{\tilde{\rho}} \frac{\partial}{\partial z} \left(\frac{\tilde{\rho} b'}{N^2} \right) = \nabla^2 \psi + f + \frac{f_0^2}{\tilde{\rho}} \frac{\partial}{\partial z} \left(\frac{\tilde{\rho}}{N^2} \frac{\partial \psi}{\partial z} \right)$$

2 Vallis (2019) Problem 5.2

- 5.2 (a) The shallow water *planetary geostrophic* equations may be derived by simply omitting ζ in the equation

$$\frac{D}{Dt} \frac{\zeta + f}{h} = 0 \quad (\text{P5.1})$$

by invoking a small Rossby number, so that ζ/f is small. We then relate the velocity field to the height field by hydrostatic balance and obtain:

$$\frac{D}{Dt} \left(\frac{f}{h} \right) = 0, \quad fu = -g \frac{\partial h}{\partial y}, \quad fv = g \frac{\partial h}{\partial x}. \quad (\text{P5.2})$$

The assumptions of hydrostatic balance and small Rossby number are the same as those used in deriving the quasi-geostrophic equations. Explain nevertheless how some of the assumptions used for quasi-geostrophy are in fact different from those used for planetary-geostrophy, and how the derivations and resulting systems differ from each other. Use any or all of the momentum and mass continuity equations, scaling, nondimensionalization and verbal explanations as needed.

- (b) Explain if and how your arguments in part (a) also apply to the stratified equations (using, for example, the Boussinesq equations or pressure coordinates).

a) The key distinction between the PG and QG equations is that in addition to small Rossby number, the QG equations also take $(L/L_d)^2 \sim 1$ scaling. But, in order to keep other terms that arise at the same order in control, the QG equations also have $\beta y/f_0 \sim Ro$ and a nonzero \mathcal{H}/H arriving at order $Ro(L/L_d)^2$.

b) The continuously stratified PG system has the buoyancy equation in place of the thickness equation, but again the $(L/L_d)^2 \gg 1$ differentiates it from the QG system where $(L/L_d)^2 \sim 1$. This makes f more variable in PG system so the $\partial w/\partial z$ term arrives at leading order in PG. In QG

the leading order geostrophic (based on f_0) velocity is nondivergent. So, the game to eliminate w arrives at a higher order equation for slight divergences, which are from: βy , ageostrophic effects from $\partial/\partial t$, ageostrophic effects from ζ , and the stretching term $f_0 \partial w / \partial z$.

3 Vallis (2019) Problem 5.4

5.4 Consider a wind stress imposed by a mesoscale cyclonic storm (in the atmosphere) given by

$$\boldsymbol{\tau} = -Ae^{-(r/\lambda)^2} (y\hat{\mathbf{i}} - x\hat{\mathbf{j}}), \quad (\text{P5.3})$$

where $r^2 = x^2 + y^2$, and A and λ are constants. Also assume constant Coriolis gradient $\beta = \partial f / \partial y$ and constant ocean depth H . In the ocean, find

- (a) the Ekman transport, (b) the vertical velocity $w_E(x, y, z)$ below the Ekman layer, (c) the northward velocity $v(x, y, z)$ below the Ekman layer and (d) indicate how you would find the westward velocity $u(x, y, z)$ below the Ekman layer.

Not yet! Next HW.

4 Vallis (2019) Problem 5.5

5.5 In an atmospheric Ekman layer on the f -plane let us write the momentum equation as

$$\mathbf{f} \times \mathbf{u} = -\nabla\phi + \frac{1}{\rho_a} \frac{\partial \boldsymbol{\tau}}{\partial z}, \quad (\text{P5.4})$$

where $\boldsymbol{\tau} = A\rho_a \partial \mathbf{u} / \partial z$ and A is a constant eddy viscosity coefficient. An independent formula for the stress at the ground is $\boldsymbol{\tau} = C\rho_a \mathbf{u}$, where C is a constant. Let us take $\rho_a = 1$, and assume that in the free atmosphere the wind is geostrophic and zonal, with $\mathbf{u}_g = U\hat{\mathbf{i}}$.

- Find an expression for the wind vector at the ground. Discuss the limits $C = 0$ and $C = \infty$. Show that when $C = 0$ the frictionally-induced vertical velocity at the top of the Ekman layer is zero.
- Find the vertically integrated horizontal mass flux caused by the boundary layer.
- When the stress on the atmosphere is $\boldsymbol{\tau}$, the stress on the ocean beneath is also $\boldsymbol{\tau}$. Why? Show how this is consistent with Newton's third law.
- Determine the direction and strength of the surface current, and the mass flux in the oceanic Ekman layer, in terms of the geostrophic wind in the atmosphere, the oceanic Ekman depth and the ratio ρ_a / ρ_o , where ρ_o is the density of the seawater. Include a figure showing the directions of the various winds and currents. How does the boundary-layer mass flux in the ocean compare to that in the atmosphere? (Assume, as needed, that the stress in the ocean may be parameterized with an eddy viscosity.)

Partial solution for (a): A useful trick in Ekman layer problems is to write the velocity as a complex number, $\hat{u} = u + iv$ and $\hat{u}_g = u_g + iv_g$. The fundamental Ekman layer equation may then be written as

$$A \frac{\partial^2 \hat{U}}{\partial z^2} = if\hat{U}, \quad (\text{P5.5})$$

where $\hat{U} = \hat{u} - \hat{u}_g$. The solution to this is

$$\hat{u} - \hat{u}_g = [\hat{u}(0) - \hat{u}_g] \exp\left[-\frac{(1+i)z}{d}\right], \quad (\text{P5.6})$$

where $d = \sqrt{2A/f}$ and the boundary condition of finiteness at infinity eliminates the exponentially growing solution. The boundary condition at $z = 0$ is $\partial \hat{u} / \partial z = (C/A)\hat{u}$; applying this gives $[\hat{u}(0) - \hat{u}_g] \exp(i\pi/4) = -Cd\hat{u}(0)/(\sqrt{2}A)$, from which we obtain $\hat{u}(0)$, and the rest of the solution follows.

Screenshot

Not yet! Next HW.

5 Vallis (2019) Problem 8.3

8.3 Following the same procedure used in Sections 8.3 and 8.7, obtain the necessary conditions for instability in the two-level quasi-geostrophic model in the case with uniform shear. Show that these conditions are consistent with the conditions for instability calculated directly with a normal-mode approach.

The equations for the 2-level or 2-layer model (Section 5.6) are:

$$\begin{aligned} \frac{D_g q_1}{Dt} &= 0, & q_1 &= \nabla^2 \psi_1 + \beta y + \frac{k_d^2}{2} (\psi_2 - \psi_1), \\ \frac{D_g q_2}{Dt} &= 0, & q_2 &= \nabla^2 \psi_2 + \beta y + \frac{k_d^2}{2} (\psi_1 - \psi_2). \end{aligned}$$

For example, a basic state resembling the Eady problem arrives via $\psi_1 = -Uy, \psi_2 = Uy$. A general plane-parallel flow yields the linearized equations

$$\begin{aligned} \frac{\partial q'_1}{\partial t} + U_1 \frac{\partial q'_1}{\partial x} + v'_1 \frac{\partial Q_1}{\partial y} &= 0, & q_1 &= \underbrace{-\frac{\partial U_1}{\partial y} + \beta y + \frac{k_d^2}{2} (\Psi_2 - \Psi_1)}_{Q_1} + \underbrace{\nabla^2 \psi'_1 + \frac{k_d^2}{2} (\psi'_2 - \psi'_1)}_{q'_1}, \\ \frac{\partial q'_2}{\partial t} + U_2 \frac{\partial q'_2}{\partial x} + v'_2 \frac{\partial Q_2}{\partial y} &= 0, & q_2 &= \underbrace{-\frac{\partial U_2}{\partial y} + \beta y + \frac{k_d^2}{2} (\Psi_1 - \Psi_2)}_{Q_2} + \underbrace{\nabla^2 \psi'_2 + \frac{k_d^2}{2} (\psi'_1 - \psi'_2)}_{q'_2}. \end{aligned}$$

Multiplying these equations by q'_1 and q'_2 yields

$$\begin{aligned} \frac{\partial}{\partial t} \frac{q_1'^2}{\frac{\partial Q_1}{\partial y}} + \frac{U_1}{\frac{\partial Q_1}{\partial y}} \frac{\partial q_1'^2}{\partial x} + 2v_1' q_1' &= 0, \\ \frac{\partial}{\partial t} \frac{q_2'^2}{\frac{\partial Q_2}{\partial y}} + \frac{U_2}{\frac{\partial Q_2}{\partial y}} \frac{\partial q_2'^2}{\partial x} + 2v_2' q_2' &= 0. \end{aligned}$$

Now, noting that

$$\begin{aligned} \iint (v_1' q_1' + v_2' q_2') dx dy &= \iint \left(v_1' \left[\cancel{\frac{\partial \psi_1'}{\partial x}} - \cancel{\frac{\partial \psi_1'}{\partial y}} + \frac{k_d^2}{2} (\psi_2' - \psi_1') \right] + v_2' \left[\cancel{\frac{\partial \psi_2'}{\partial x}} - \cancel{\frac{\partial \psi_2'}{\partial y}} + \frac{k_d^2}{2} (\psi_1' - \psi_2') \right] \right) dx dy \\ &= \iint \frac{k_d^2}{2} \left(\frac{\partial \psi_1'}{\partial x} [(\psi_2' - \psi_1')] + \frac{\partial \psi_2'}{\partial x} [(\psi_1' - \psi_2')] \right) dx dy \\ &= \iint \frac{k_d^2}{4} \left(-\frac{\partial}{\partial x} (\psi_1' - \psi_2')^2 \right) dx dy = 0 \end{aligned}$$

So,

$$\frac{d}{dt} \left\{ \iint \left[\frac{q_1'^2}{\frac{\partial Q_1}{\partial y}} + \frac{q_2'^2}{\frac{\partial Q_2}{\partial y}} \right] dx dy \right\} = 0$$

Thus, the necessary condition for instability in the 2-level model is: Q_y must change sign somewhere in the domain, which may either be by variation of Q_y in y or by having $\frac{\partial Q_1}{\partial y}$ and $\frac{\partial Q_2}{\partial y}$ be of different sign.

References

Vallis, G. K. (2019). *Essentials of Atmospheric and Oceanic Dynamics*. Cambridge University Press.