Spring 2020 GEOL1820
Homework 4, due Monday, April 6, 9AM

1 Vallis (2019) Problem 5.4

5.4 Consider a wind stress imposed by a mesoscale cyclonic storm (in the atmosphere) given by

$$\boldsymbol{\tau} = -A\,\mathrm{e}^{-(r/\lambda)^2}(y\,\hat{\mathbf{i}} - x\,\hat{\mathbf{j}}), \tag{P5.3}$$

where $r^2 = x^2 + y^2$, and A and λ are constants. Also assume constant Coriolis gradient $\beta = \partial f/\partial y$ and constant ocean depth H. In the ocean, find

(a) the Ekman transport, (b) the vertical velocity $w_E(x, y, z)$ below the Ekman layer, (c) the northward velocity $v(x, y, z)$ below the Ekman layer and (d) indicate how you would find the westward velocity $u(x, y, z)$ below the Ekman layer.

2 Vallis (2019) Problem 5.5

5.5 In an atmospheric Ekman layer on the f-plane let us write the momentum equation as

$$\boldsymbol{f} \times \boldsymbol{u} = -\nabla\phi + \frac{1}{\rho_a}\frac{\partial \boldsymbol{\tau}}{\partial z}, \tag{P5.4}$$

where $\boldsymbol{\tau} = A\rho_a \partial \boldsymbol{u}/\partial z$ and A is a constant eddy viscosity coefficient. An *independent* formula for the stress at the ground is $\boldsymbol{\tau} = C\rho_a \boldsymbol{u}$, where C is a constant. Let us take $\rho_a = 1$, and assume that in the free atmosphere the wind is geostrophic and zonal, with $\boldsymbol{u}_g = U\hat{\mathbf{i}}$.

(a) Find an expression for the wind vector at the ground. Discuss the limits $C = 0$ and $C = \infty$. Show that when $C = 0$ the frictionally-induced vertical velocity at the top of the Ekman layer is zero.

(b) Find the vertically integrated horizontal mass flux caused by the boundary layer.

(c) When the stress on the atmosphere is $\boldsymbol{\tau}$, the stress on the ocean beneath is also $\boldsymbol{\tau}$. Why? Show how this is consistent with Newton's third law.

(d) Determine the direction and strength of the surface current, and the mass flux in the oceanic Ekman layer, in terms of the geostrophic wind in the atmosphere, the oceanic Ekman depth and the ratio ρ_a/ρ_o, where ρ_o is the density of the seawater. Include a figure showing the directions of the various winds and currents. How does the boundary-layer mass flux in the ocean compare to that in the atmosphere? (Assume, as needed, that the stress in the ocean may be parameterized with an eddy viscosity.)

Partial solution for (a): A useful trick in Ekman layer problems is to write the velocity as a complex number, $\hat{u} = u + \mathrm{i}v$ and $\hat{u}_g = u_g + \mathrm{i}v_g$. The fundamental Ekman layer equation may then be written as

$$A\frac{\partial^2 \widehat{U}}{\partial z^2} = \mathrm{i} f \widehat{U}, \tag{P5.5}$$

where $\widehat{U} = \hat{u} - \hat{u}_g$. The solution to this is

$$\hat{u} - \hat{u}_g = [\hat{u}(0) - \hat{u}_g] \exp\left[-\frac{(1+\mathrm{i})z}{d}\right], \tag{P5.6}$$

where $d = \sqrt{2A/f}$ and the boundary condition of finiteness at infinity eliminates the exponentially growing solution. The boundary condition at $z = 0$ is $\partial\hat{u}/\partial z = (C/A)\hat{u}$; applying this gives $[\hat{u}(0) - \hat{u}_g]\exp(\mathrm{i}\pi/4) = -Cd\hat{u}(0)/(\sqrt{2}A)$, from which we obtain $\hat{u}(0)$, and the rest of the solution follows.

3 Vallis (2019) Problem 10.1

10.1 *Predictability.* The eddy turnover time of three-dimensional turbulence at a wavenumber k is given by $\tau_k = \varepsilon^{-1/3}k^{-2/3}$, and that of two-dimensional turbulence by $\tau_k = \eta^{-1/3}$ where k is the wavenumber and ε and η are constants (the energy and enstrophy cascade rates, respectively). Suppose that in weather prediction the error is confined to small scales and that the time taken for the error to contaminate the next largest scale (in a logarithmic sense) is τ_k, so that the time taken for an error at a small scale k_s to reach the large scales k_l is given by

$$T = \int_{k_s}^{k_l} \tau_k \, \mathrm{d}k. \qquad \text{(P10.1)}$$

If the inertial range extends indefinitely show that this time is infinite for classical two-dimensional turbulence and finite for three-dimensional turbulence, and discuss the implications for weather predictability and weather forecasting. If the atmosphere is two-dimensional down to a scale of 50 km, estimate a limit to weather predictability (e.g., a timescale in days), making sensible (and clearly stated) assumptions about the magnitude of the flow.

4 Thorpe (2007) Problem 1.2

P1.2 (E) The criterion for turbulence. Reynolds' experiment shows that turbulence with eddies of size comparable to the tube radius develops when the Reynolds number, *Re*, exceeds a critical value. The mean depth of the Irish Sea is about 60 m and the tidal currents are typically 0.1–1 m s^{-1}. With only this information and assuming that the critical Reynolds number for oceanic flows is of order 10^4, should the tidal flow in the Irish Sea be laminar or should it be turbulent, probably with some eddies of size comparable to the water depth? • The estimate of *Re* was made by G. I. Taylor, 1919, and used to dismiss earlier calculations of the dissipation of tidal energy in the Irish Sea carried out assuming only molecular viscosity in a laminar flow over the seabed and in the water column.

NOTE: Thorpe (2007) is accessible at `https://login.revproxy.brown.edu/login?url=http://search.ebscohost.com/login.aspx?direct=true&scope=site&db=e000xna&AN=304600`

5 Thorpe (2007) Problem 1.9

P1.9 (M) The energy needed to mix a stratified region. What is the minimum energy required to reduce an initially uniform density gradient with buoyancy frequency N_0 to a final state with frequency $N < N_0$, over a depth of $2h$? (An application of this calculation is found in the description by Sundermeyer *et al*., 2005, of the horizontal diffusion of dye in the ocean.)

6 Thorpe (2007) Problem 2.5

P1.2 (E) The criterion for turbulence. Reynolds' experiment shows that turbulence with eddies of size comparable to the tube radius develops when the Reynolds number, *Re*, exceeds a critical value. The mean depth of the Irish Sea is about 60 m and the tidal currents are typically 0.1–1 m s^{-1}. With only this information and assuming that the critical Reynolds number for oceanic flows is of order 10^4, should the tidal flow in the Irish Sea be laminar or should it be turbulent, probably with some eddies of size comparable to the water depth? • The estimate of *Re* was made by G. I. Taylor, 1919, and used to dismiss earlier calculations of the dissipation of tidal energy in the Irish Sea carried out assuming only molecular viscosity in a laminar flow over the seabed and in the water column.

7 Thorpe (2007) Problem 2.6

ment. P2.6 (D) The Lagrangian spectrum. If the frequency spectrum of the vertical component of velocity, $\Phi_w(\sigma)$, measured in a Lagrangian frame of reference in stratified turbulent flow depends, in an inertial range, only on the frequency, σ, and the rate of dissipation of turbulent kinetic energy, but – in the range dominated by inertial rather than buoyancy forces – is independent of buoyancy frequency, N, find how Φ_w varies with σ. Use Fig. 2.6 to determine any unknown constant. P2.7 (M) Turbulent

8 Thorpe (2007) Problem 3.5

P3.5 (E) The Kolmogorov scale in a boundary layer. How does the Kolmogorov length scale, l_K, vary with distance from the seabed, with the friction velocity and with the stress within a law-of-the-wall constant-stress layer if $C_D = 2.5 \times 10^{-3}$? Estimate l_K at a height of 1 m from the bed in tidal flows of 0.2 and 1 m s^{-1}.

9 Final Project Proposal/Abstract

Please include a 1 page proposal for your final paper, following the instructions here: http://www.geo.brown.edu/research/Fox-Kemper/classes/GEOL1820_20/notes/2020_FinalPaper.pdf.

References

Thorpe, S. A. (2007). *An introduction to ocean turbulence*. Cambridge University Press, Cambridge.

Vallis, G. K. (2019). *Essentials of Atmospheric and Oceanic Dynamics*. Cambridge University Press.