Spring 2020 GEOL1820
Homework 4, due Monday, April 6, 9AM

1 Vallis (2019) Problem 5.4

5.4 Consider a wind stress imposed by a mesoscale cyclonic storm (in the atmosphere) given by

$$\boldsymbol{\tau} = -Ae^{-(r/\lambda)^2}(y\,\hat{\mathbf{i}} - x\,\hat{\mathbf{j}}), \tag{P5.3}$$

where $r^2 = x^2 + y^2$, and A and λ are constants. Also assume constant Coriolis gradient $\beta = \partial f/\partial y$ and constant ocean depth H. In the ocean, find

(a) the Ekman transport, (b) the vertical velocity $w_E(x, y, z)$ below the Ekman layer, (c) the northward velocity $v(x, y, z)$ below the Ekman layer and (d) indicate how you would find the westward velocity $u(x, y, z)$ below the Ekman layer.

a) This is cleanly found from Section 14.2 of Vallis (2019), which is very similar to equations from Wyngaard (2010). These are the steady boundary layer equations with an imposed stress.

$$-fv = -\frac{\partial \phi}{\partial x} + \frac{1}{\rho_0}\frac{\partial \tau^x}{\partial z} \tag{1}$$

$$fu = -\frac{\partial \phi}{\partial y} + \frac{1}{\rho_0}\frac{\partial \tau^y}{\partial z} \tag{2}$$

Or, writing out the pressure gradients in terms of their geostrophic velocities,

$$f(v_g - v) = -fv_a = \frac{1}{\rho_0}\frac{\partial \tau^x}{\partial z} \tag{3}$$

$$f(u - u_g) = fu_a = \frac{1}{\rho_0}\frac{\partial \tau^y}{\partial z} \tag{4}$$

Integrating in the vertical we find the Ekman transport:

$$(V - V_g) = \int_{-h}^{0} v_a \,\mathrm{d}z = -\frac{1}{\rho_0 f}\tau^x \tag{5}$$

$$(U - U_g) = \int_{-h}^{0} u_a \,\mathrm{d}z = \frac{1}{\rho_0 f}\tau^y \tag{6}$$

b) Find the vertical velocity at the bottom of the Ekman layer. We consider incompressibility and then integrate over the Ekman layer:

$$\frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \tag{7}$$

$$w(-h) = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \tag{8}$$

$$= \frac{\partial}{\partial x}\frac{\tau^y}{\rho_0 f} - \frac{\partial}{\partial y}\frac{\tau^x}{\rho_0 f} \tag{9}$$

$$= \frac{2A}{\rho_0 f}e^{-(r/\lambda)^2} - \frac{2Ar^2}{\rho_0 f \lambda^2}e^{-(r/\lambda)^2} - \frac{Ay\beta}{\rho_0 f^2}e^{-(r/\lambda)^2} \tag{10}$$

c) The northward velocity below the Ekman layer can be found from integrating the vorticity budgets, across the layers below the surface layer and ignoring bottom drag, or

$$f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta v = 0 - \cancelto{0}{\frac{1}{\rho_0}\left(\frac{\partial^2 \tau^x}{\partial z \partial y} - \frac{\partial^2 \tau^y}{\partial z \partial x}\right)} \tag{11}$$

$$\int_{-H}^{-h} v \, \mathrm{d}z = \frac{1}{\beta}\left[\frac{\partial}{\partial x}\frac{\tau^y}{\rho_0 f} - \frac{\partial}{\partial y}\frac{\tau^x}{\rho_0 f}\right] \tag{12}$$

$$\int_{-H}^{-h} v \, \mathrm{d}z = \frac{1}{\beta}\left[\frac{2A}{\rho_0}e^{-(r/\lambda)^2} - \frac{2Ar^2}{\rho_0 \lambda^2}e^{-(r/\lambda)^2} - \frac{Ay\beta}{\rho_0 f}e^{-(r/\lambda)^2}\right] \tag{13}$$

If the flow is barotropic below the boundary layer, then it will be independent of depth for all depths below $-h$, or

$$v = \frac{1}{\beta(H-h)}\left[\frac{2A}{\rho_0}e^{-(r/\lambda)^2} - \frac{2Ar^2}{\rho_0 \lambda^2}e^{-(r/\lambda)^2} - \frac{Ay\beta}{\rho_0 f}e^{-(r/\lambda)^2}\right] \tag{14}$$

d) To find the u velocity associated with this v, one can find the pressure field associated with the v velocity (i.e., the streamfunction), and then use the continuity equation to find u.

2 Vallis (2019) Problem 5.5

5.5 In an atmospheric Ekman layer on the f-plane let us write the momentum equation as

$$\boldsymbol{f} \times \boldsymbol{u} = -\nabla\phi + \frac{1}{\rho_a}\frac{\partial \boldsymbol{\tau}}{\partial z}, \tag{P5.4}$$

where $\boldsymbol{\tau} = A\rho_a \partial \boldsymbol{u}/\partial z$ and A is a constant eddy viscosity coefficient. An *independent* formula for the stress at the ground is $\boldsymbol{\tau} = C\rho_a \boldsymbol{u}$, where C is a constant. Let us take $\rho_a = 1$, and assume that in the free atmosphere the wind is geostrophic and zonal, with $\boldsymbol{u}_g = U\hat{\mathbf{i}}$.

(a) Find an expression for the wind vector at the ground. Discuss the limits $C = 0$ and $C = \infty$. Show that when $C = 0$ the frictionally-induced vertical velocity at the top of the Ekman layer is zero.

(b) Find the vertically integrated horizontal mass flux caused by the boundary layer.

(c) When the stress on the atmosphere is $\boldsymbol{\tau}$, the stress on the ocean beneath is also $\boldsymbol{\tau}$. Why? Show how this is consistent with Newton's third law.

(d) Determine the direction and strength of the surface current, and the mass flux in the oceanic Ekman layer, in terms of the geostrophic wind in the atmosphere, the oceanic Ekman depth and the ratio ρ_a/ρ_o, where ρ_o is the density of the seawater. Include a figure showing the directions of the various winds and currents. How does the boundary-layer mass flux in the ocean compare to that in the atmosphere? (Assume, as needed, that the stress in the ocean may be parameterized with an eddy viscosity.)

Partial solution for (a): A useful trick in Ekman layer problems is to write the velocity as a complex number, $\hat{u} = u + \mathrm{i}v$ and $\hat{u}_g = u_g + \mathrm{i}v_g$. The fundamental Ekman layer equation may then be written as

$$A\frac{\partial^2 \widehat{U}}{\partial z^2} = \mathrm{i} f \widehat{U}, \tag{P5.5}$$

where $\widehat{U} = \hat{u} - \hat{u}_g$. The solution to this is

$$\hat{u} - \hat{u}_g = [\hat{u}(0) - \hat{u}_g]\exp\left[-\frac{(1+\mathrm{i})z}{d}\right], \tag{P5.6}$$

where $d = \sqrt{2A/f}$ and the boundary condition of finiteness at infinity eliminates the exponentially growing solution. The boundary condition at $z = 0$ is $\partial\hat{u}/\partial z = (C/A)\hat{u}$; applying this gives $[\hat{u}(0) - \hat{u}_g]\exp(\mathrm{i}\pi/4) = -Cd\hat{u}(0)/(\sqrt{2}A)$, from which we obtain $\hat{u}(0)$, and the rest of the solution follows.

We begin by organizing the equations somewhat, with no baroclinicity (geostrophic velocity/pressure gradient independent of height), then

$$f(u-u_g) = -\frac{\partial \tau^y/\rho_0}{\partial z} = A\frac{\partial^2}{\partial z^2}v \tag{15}$$

$$f(v-v_g) = \frac{\partial \tau^x/\rho_0}{\partial z} = -A\frac{\partial^2}{\partial z^2}u \tag{16}$$

With the hint, this this becomes

$$\hat{u}-\hat{u}_g = \left[\hat{u}(0)-\hat{u}_g\right] e^{\frac{-(1+i)z}{d}} \tag{17}$$

$$\frac{\partial}{\partial z}\hat{u} = \frac{-(1+i)}{d}\left[\hat{u}(0)-\hat{u}_g\right] = \frac{C}{A}\hat{u}(0) \tag{18}$$

$$d = \sqrt{\frac{2A}{f}} \tag{19}$$

a) Find $\hat{u}(0)$.

$$\hat{u}(0)\left(1+\frac{Cd}{A(1+i)}\right) = U_g \tag{20}$$

$$\hat{u}(0) = U_g \frac{2+(1+i)\frac{Cd}{A}}{1+\left(1+\frac{Cd}{A}\right)^2} \tag{21}$$

$$\lim_{C\to 0}\hat{u}(0) = U_g \tag{22}$$

$$\lim_{C\to\infty}\hat{u}(0) = \lim_{C\to\infty} U_g\frac{(1+i)A}{Cd} = 0 \tag{23}$$

So the drag coefficient strongly affects the flow at the ground. Furthermore, when $C=0$, then $\hat{u}(0)=U_g$, which means that the vertical velocity is also zero.
b) The vertical integral of the mass flux is

$$\int_0^\infty \hat{u}-\hat{u}_g\,\mathrm{d}z = \frac{d}{1+i}\left[\hat{u}(0)-\hat{u}_g\right] = d\left[\hat{u}(0)-\hat{u}_g\right]\left(\frac{1}{2}-\frac{i}{2}\right)\left(\frac{2+(1+i)\frac{Cd}{A}}{1+\left(1+\frac{Cd}{A}\right)^2}-1\right) \tag{24}$$

c) The stress constitutes a vertical transport of momentum. The stress on one fluid must be exerted on the other if there is to be no accumulation of momentum. In fact, there can be an accumulation of momentum at the interface if it is carried away by interfacial waves.
d) The ocean has the same momentum balance as the atmosphere, but we need to retain the other root of the exponential so that the solution decays as z goes to minus infinity.

$$\hat{u}_o-(\hat{u}_g)_o = \left[\hat{u}_o(0)-(\hat{u}_g)_o\right] e^{\frac{(1+i)z}{d_o}} \tag{25}$$

$$\frac{\partial}{\partial z}\hat{u} = \frac{-(1+i)}{d_o}\left[\hat{u}_o(0)-(\hat{u}_g)_o\right] = \frac{C_o}{A_o}\hat{u}_o(0) = \frac{C_a\rho_a\hat{u}_a(0)}{A_o\rho_o} = \frac{C_a\rho_a U_g}{A_o\rho_o}\frac{2+(1+i)\frac{C_a d_a}{A_a}}{1+\left(1+\frac{C_a d_a}{A_a}\right)^2} \tag{26}$$

$$d_o = \sqrt{\frac{2A_o}{f}} \tag{27}$$

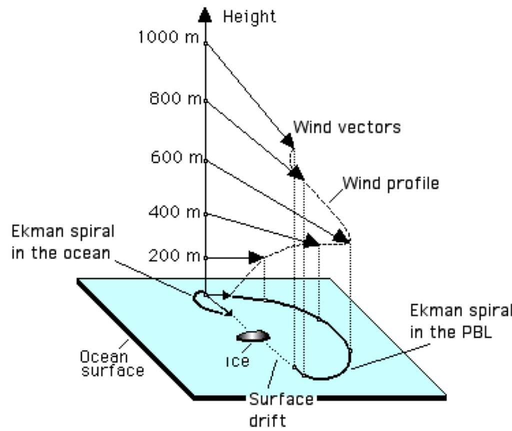

Figure 1: Ekman spiral directions (courtesy: Z. Sorbjan)

The oceanic flow is to the right of the surface wind, and the Ekman spirals are in opposing directions.

The vertical integral of the momentum flux doesn't depend on the eddy viscosity, as the vertical integral of

$$\rho f(u - u_g) = -\frac{\partial \tau^y}{\partial z} \tag{28}$$

$$\rho f(v - v_g) = \frac{\partial \tau^x}{\partial z} \tag{29}$$

depends only on the density times velocity (momentum/mass transport) and the stress. Going from high in the atmosphere to deep in the ocean, where the stresses vanish, we can see that there can be no integrated deviation from the net geostrophic mass flux in added together in both fluids. Noting that the stress on one fluid equals the stress on the other, we can see that the Ekman mass flux in the atmosphere equals the Ekman mass flux in the ocean.

3 Vallis (2019) Problem 10.1

10.1 *Predictability.* The eddy turnover time of three-dimensional turbulence at a wavenumber k is given by $\tau_k = \varepsilon^{-1/3} k^{-2/3}$, and that of two-dimensional turbulence by $\tau_k = \eta^{-1/3}$ where k is the wavenumber and ε and η are constants (the energy and enstrophy cascade rates, respectively). Suppose that in weather prediction the error is confined to small scales and that the time taken for the error to contaminate the next largest scale (in a logarithmic sense) is τ_k, so that the time taken for an error at a small scale k_s to reach the large scales k_l is given by

$$T = \int_{k_s}^{k_l} \tau_k \, \mathrm{d}k. \tag{P10.1}$$

If the inertial range extends indefinitely show that this time is infinite for classical two-dimensional turbulence and finite for three-dimensional turbulence, and discuss the implications for weather predictability and weather forecasting. If the atmosphere is two-dimensional down to a scale of 50 km, estimate a limit to weather predictability (e.g., a timescale in days), making sensible (and clearly stated) assumptions about the magnitude of the flow.

This problem had issues in the statement. I have contacted Vallis to get his take, and have updated to his response.

Let's restate the problem as the predictability timescale depending on the average time, and k_s being the small scales (i.e., large wavenumbers) and k_l being the large scales (small wavenumbers). Also, the differential should go with the logarithm of k, which is because the way to think about time here is how long it takes for an error at small scales to pass to a larger scale that is a *multiple* of the original scale, e.g., wavenumber $k/2$, not a fixed additive $\mathrm{d}k$ larger. Vallis recommended reading a few papers as well (Vallis, 1985; Lorenz, 1969b,a; Kraichnan, 1971; Lilly, 1972). Lorenz (1969b) is particularly helpful.

$$T = \int_{\ln k_l}^{\ln k_s} \tau_k \, \mathrm{d}(\ln k) \tag{30}$$

The 2D case is then:

$$T_{2D} = \lim_{k_s \to \infty} \int_{\ln k_l}^{\ln k_s} \eta^{-1/3} \, \mathrm{d}(\ln k) = \lim_{k_s \to \infty} \int_{k_l}^{k_s} \eta^{-1/3} \frac{\mathrm{d}k}{k} = \lim_{k_s \to \infty} \eta^{-1/3} \ln(k_s/k_l) = \infty \tag{31}$$

The 3D case is then:

$$T_{3D} = \lim_{k_s \to \infty} \int_{k_l}^{k_s} \epsilon^{-1/3} k^{-2/3} \frac{\mathrm{d}k}{k} = \lim_{k_s \to \infty} \frac{3}{2} \epsilon^{-1/3} \left[k_l^{-2/3} - k_s^{-2/3} \right] = \frac{3}{2} \epsilon^{-1/3} k_l^{-2/3} \tag{32}$$

The case where it is 3D on scales smaller than $k_b = 2\pi/50$ km and 2D on larger scales is

$$\begin{aligned} T_{mix} &= \lim_{k_s \to \infty} \int_{k_l}^{k_b} \eta^{-1/3} \frac{\mathrm{d}k}{k} + \lim_{k_s \to \infty} \int_{k_b}^{k_s} \epsilon^{-1/3} k^{-2/3} \frac{\mathrm{d}k}{k} && (33) \\ &= \lim_{k_s \to \infty} \left[\eta^{-1/3} \ln(k_b/k_l) + \frac{3}{2} \epsilon^{-1/3} \left(k_b^{-2/3} - k_s^{-2/3} \right) \right] && (34) \\ &= \left[\eta^{-1/3} \ln(k_b/k_l) + \frac{3}{2} \epsilon^{-1/3} \left(k_b^{-2/3} \right) \right] && (35) \end{aligned}$$

Let's estimate the enstrophy and energy cascade rates. One method would be to use observations, such as the important Nastrom and Gage (1985) ones. Using this data together with the enstrophy cascade $E(k) \sim \eta^{2/3}k^{-3}$ and energy cascade scalings $E(k) \sim \epsilon^{2/3}k^{-5/3}$ where k=$2\pi/(50km) \approx 10^{-4}/m$, and assuming a constant of proportionality (the Kraichnan-Kolmogorov constants) between these scalings of spectra and the spectra themselves, we find that $E(k) \approx 10^5 m^3/s^2$ implies $\eta \approx \left[(10^5 m^3/s^2)(10^{-4}/m)^3\right]^{3/2} = 3 \cdot 10^{-11} s^{-3}$, and since the breakpoint between 2D and 3D occurs there, likewise $\epsilon \approx \left[(10^5 m^3/s^2)(10^{-4}/m)^{5/3}\right]^{3/2} = 3 \cdot 10^{-3} m^2 s^{-3}$. Plugging in, this implies

$$T_{mix} = \left[\eta^{-1/3}\ln(k_b/k_l) + \frac{3}{2}\epsilon^{-1/3}\left(k_b^{-2/3}\right)\right] \tag{36}$$

$$\approx \left[\eta^{-1/3}\ln(10,000/50) + \frac{3}{2}\epsilon^{-1/3}\left(k_b^{-2/3}\right)\right] \tag{37}$$

$$= 0.0372 \cdot [5 + 1.5] \text{ days} \tag{38}$$

$$= 0.24 \text{ days} \tag{39}$$

That value is a bit short, probably related to the issues surrounding the estimation of the η, ϵ and the neglect of the Kolmogorov-Kraichnan constants. Note that the contribution over the 2D part of the integral is larger, and that the largest eddies are taken to be roughly 10,000 km.
An alternative, is just to calculate the timescales and sum them up. As we saw above, it takes about 7-8 steps of a factor of 2, i.e., $2^7 = 128 < 10,000/50 < 256 = 2^8$. If we take the eddy turnover time as the timescale for each step, then we can get

$$T_{mix} = \tau_{6400km} + \tau_{3200km} + \tau_{1600km} + \tau_{800km} + \tau_{400km} + \tau_{200km} + \tau_{100km} + \tau_{50km} \tag{40}$$

$$T_{mix} = 8\tau_{50km} \approx 8 \text{ days} \tag{41}$$

Where we have taken advantage of the fact that eddy turnover time is independent of scale in the 2D case, and that a typical eddy might have Rossby number of 1. Lorenz (1969b) finds values from 2.5 to 5.6 days at this scale, by a much more elaborate method.

RETRACTED OLD VERSION: This problem has issues in the statement in *Essentials*. I have contacted Vallis to get his take, but this was my answer based on his old version.
Let's restate the problem as the predictability timescale depending on the average time, and k_s being the small scales (i.e., large wavenumbers) and k_l being the large scales (small wavenumbers):

$$T = \frac{1}{k_s - k_l}\int_{k_l}^{k_s} \tau_k \, \mathrm{d}k \tag{42}$$

The 2D case is then:

$$T_{2D} = \lim_{k_s \to \infty} \frac{1}{k_s - k_l}\int_{k_l}^{k_s} \eta^{1/3} \, \mathrm{d}k = \eta^{1/3} \tag{43}$$

The 3D case is then:

$$T_{3D} = \lim_{k_s \to \infty} \frac{1}{k_s - k_l}\int_{k_l}^{k_s} \epsilon^{-1/3}k^{-2/3} \, \mathrm{d}k = \lim_{k_s \to \infty} \epsilon^{-1/3}\frac{k_s^{1/3} - k_l^{1/3}}{k_s - k_l} = \lim_{k_s \to \infty} \epsilon^{-1/3}\, k_s^{-2/3} = 0 \tag{44}$$

The case where it is 3D on scales larger than $k_b = 2\pi/50$ km is

$$T_{mix} = \lim_{k_s\to\infty} \frac{1}{k_s - k_l}\int_{k_l}^{k_b} \eta^{1/3}\,\mathrm{d}k + \lim_{k_s\to\infty} \frac{1}{k_s - k_l}\int_{k_b}^{k_s} \epsilon^{-1/3}k^{-2/3}\,\mathrm{d}k \tag{45}$$

$$= \lim_{k_s\to\infty}\left[\eta^{1/3}\frac{k_b - k_l}{k_s - k_l} + \epsilon^{-1/3}\frac{k_s^{1/3} - k_b^{1/3}}{k_s - k_l}\right] \tag{46}$$

$$= 0 \tag{47}$$

4 Thorpe (2007) Problem 1.2

P1.2 (E) The criterion for turbulence. Reynolds' experiment shows that turbulence with eddies of size comparable to the tube radius develops when the Reynolds number, *Re*, exceeds a critical value. The mean depth of the Irish Sea is about 60 m and the tidal currents are typically 0.1–1 m s^{-1}. With only this information and assuming that the critical Reynolds number for oceanic flows is of order 10^4, should the tidal flow in the Irish Sea be laminar or should it be turbulent, probably with some eddies of size comparable to the water depth? • The estimate of *Re* was made by G. I. Taylor, 1919, and used to dismiss earlier calculations of the dissipation of tidal energy in the Irish Sea carried out assuming only molecular viscosity in a laminar flow over the seabed and in the water column.

NOTE: Thorpe (2007) is accessible at https://login.revproxy.brown.edu/login?url=http://search.ebscohost.com/login.aspx?direct=true&scope=site&db=e000xna&AN=304600

The Reynolds number criterion for pipe flow is

$$Re > Re_c = \mathcal{O}(10^4) \tag{48}$$

Interpreting this for the Irish Sea with $d \approx 60m, U \approx 0.1 \to 1.0m/s, \nu \approx 10^{-6}m^2/s$, we find

$$Re \sim \frac{Ud}{\nu} = \frac{6m^2/s \to 60m^2/s}{10^{-6}m^2/s} = 6\cdot 10^6 \to 6\cdot 10^7 \gg 10^4 \tag{49}$$

Thus, the Irish Sea is expected to be highly turbulent with a Reynolds number more than 100x the critical one.

5 Thorpe (2007) Problem 1.9

P1.9 (M) The energy needed to mix a stratified region. What is the minimum energy required to reduce an initially uniform density gradient with buoyancy frequency N_0 to a final state with frequency $N < N_0$, over a depth of $2h$? (An application of this calculation is found in the description by Sundermeyer *et al.*, 2005, of the horizontal diffusion of dye in the ocean.)

We consider the problem for the initial density ρ_i and final density ρ_f. They are both related to stratification by a differential equation

$$N_0^2 = \frac{-g}{\rho_0}\frac{\partial \rho_i}{\partial z}, \tag{50}$$

$$N^2 = \frac{-g}{\rho_0}\frac{\partial \rho_f}{\partial z} \tag{51}$$

which are solved for constant stratification by

$$\rho_i = \frac{-N_0^2\rho_0}{g}z + c_i, \tag{52}$$

$$\rho_f = \frac{-N^2\rho_0}{g}z + c_f. \tag{53}$$

If we enforce conservation of mass over the layer from $-h$ to h,

$$2hc_i = \int_{-h}^{h}\left(\frac{-N_0^2\rho_0}{g}z + c_i\right)\mathrm{d}z = \int_{-h}^{h}\rho_i\,\mathrm{d}z = \int_{-h}^{h}\rho_f\,\mathrm{d}z = \int_{-h}^{h}\left(\frac{-N^2\rho_0}{g}z + c_f\right)\mathrm{d}z = 2hc_f \tag{54}$$

Which tells us that $c_i = c_f = c$. Now, we consider the potential energy per unit volume ($\rho g h$) change from the initial state to the final, which gives the potential energy change as

$$A\int_{-h}^{h}\rho_i g z\,\mathrm{d}z = A\int_{-h}^{h}\left(\frac{-N^2\rho_0}{g}z + c\right)gz\,\mathrm{d}z = \frac{-2AN^2\rho_0 h^3}{3} \tag{55}$$

$$A\int_{-h}^{h}\rho_f g z\,\mathrm{d}z = A\int_{-h}^{h}\left(\frac{-N_0^2\rho_0}{g}z + c\right)gz\,\mathrm{d}z = \frac{-2AN_0^2\rho_0 h^3}{3} \tag{56}$$

with A as the area. Thus, the rise in potential energy, which is the minimum possible expended for the mixing, is

$$\frac{2(N_0^2 - N^2)A\rho_0 h^3}{3} \tag{57}$$

6 Thorpe (2007) Problem 2.5

P2.5 (M) The minus five-thirds spectrum. Derive (2.15) using a dimensional argument. P2.6 (D) The Lagrangian spectrum. If the frequency spectrum of the vertical

If we use angle brackets to denote the units of a particular quantity, then

$$[\Phi(k)] = \left[\frac{\text{Energy}}{m\ \text{wavenumber}}\right] = \frac{L^3}{T^2} \tag{58}$$

$$[\epsilon] = \left[\frac{\text{Energy}}{m\ \text{time}}\right] = \frac{L^2}{T^3} \tag{59}$$

$$[k] = \frac{1}{L} \tag{60}$$

$$[\Phi(k)] = \left[q\epsilon^a k^b\right] = \left(\frac{L^2}{T^3}\right)^a\left(\frac{1}{L}\right)^b = \frac{L^3}{T^2} \tag{61}$$

This is only satisfied if $a = 2/3$ and $b = -5/3$.

7 Thorpe (2007) Problem 2.6

ment. P2.6 (D) The Lagrangian spectrum. If the frequency spectrum of the vertical component of velocity, $\Phi_w(\sigma)$, measured in a Lagrangian frame of reference in stratified turbulent flow depends, in an inertial range, only on the frequency, σ, and the rate of dissipation of turbulent kinetic energy, but – in the range dominated by inertial rather than buoyancy forces – is independent of buoyancy frequency, N, find how Φ_w varies with σ. Use Fig. 2.6 to determine any unknown constant. P2.7 (M) Turbulent

If we use angle brackets to denote the units of a particular quantity, then

$$\left[\Phi_\omega(\omega)\right] = \left[\frac{\text{Energy}}{m \text{ frequency}}\right] = \frac{L^2}{T} \tag{62}$$

$$[\epsilon] = \left[\frac{\text{Energy}}{m \text{ time}}\right] = \frac{L^2}{T^3} \tag{63}$$

$$[\sigma] = T^{-1} \tag{64}$$

$$\left[\Phi_\omega(\omega)\right] = [\epsilon]\left[\sigma^{-2}\right] \tag{65}$$

$$\Phi_\omega(\omega) = c\epsilon\sigma^{-2} \tag{66}$$

Examining Fig. 2.6, we see that when $\log(\sigma/N) = 0$, i.e., $\sigma = N$, then $\log(\Phi/(\epsilon N^2)) = 0$, so $\Phi = \epsilon N^2$. Thus,

$$\Phi_\omega(\omega) = \epsilon\sigma^{-2} \qquad \text{when} \qquad \sigma \gg N, \tag{67}$$

$$\Phi_\omega(\omega) = \epsilon N^{-2} \qquad \text{when} \qquad \sigma \ll N. \tag{68}$$

Where the last relation is not obvious from dimensional analysis alone, but seems clear from Fig. 2.6 which shows that the spectrum is not dependent on σ for small values.

8 Thorpe (2007) Problem 3.5

P3.5 (E) The Kolmogorov scale in a boundary layer. How does the Kolmogorov length scale, l_K, vary with distance from the seabed, with the friction velocity and with the stress within a law-of-the-wall constant-stress layer if $C_D = 2.5 \times 10^{-3}$? Estimate l_K at a height of 1 m from the bed in tidal flows of 0.2 and 1 m s^{-1}.

Revisiting the log-layer relations in Thorpe, we find

$$\epsilon = \frac{\tau}{\rho_0}\frac{dU}{dz} = \frac{\tau}{\rho_0}\frac{u_*}{kz} = \frac{u_*^3}{kz}, \tag{69}$$

$$l_K = \left(\frac{\nu^3}{\epsilon}\right)^{1/4} = \left(\frac{\nu^3 kz}{u_*^3}\right)^{1/4} = \left(\frac{\nu}{u_*}\right)^{3/4}(kz)^{1/4} \tag{70}$$

which gives the dependence of the Kolmogorov scale on z. If we want to estimate the values, we need a few more pieces of info

$$u_*^2 = C_d U^2 = 2.5 \cdot 10^{-3} U^2 = \left(\frac{1}{20} U\right)^2 \tag{71}$$

$$u_* = \sqrt{C_d} U = \frac{U}{20} \tag{72}$$

$$k = 0.41 \tag{73}$$

$$\nu = 1 \cdot 10^{-6} m^2/s \tag{74}$$

So, if $U = 0.2m/s$, then

$$U = 0.2m/s \tag{75}$$

$$u_* = 0.01m/s \tag{76}$$

$$l_K = \left(\frac{\nu}{u_*}\right)^{3/4} (0.41 \times 1m)^{1/4} \approx 8 \cdot 10^{-4} m \tag{77}$$

So, if $U = 1.0m/s$, then

$$U = 1.0m/s \tag{78}$$

$$u_* = 0.05m/s \tag{79}$$

$$l_K = \left(\frac{\nu}{u_*}\right)^{3/4} (0.41 \times 1m)^{1/4} \approx 2 \cdot 10^{-4} m \tag{80}$$

9 Final Project Proposal/Abstract

Please include a 1 page proposal for your final paper, following the instructions here: `http://www.geo.brown.edu/research/Fox-Kemper/classes/GEOL1820_20/notes/2020_FinalPaper.pdf`.

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