Spring 2020 GEOL1820
Homework 5 (Last One!), due Monday, April 27, 9AM

1 Wyngaard (2010) Problem 8.2

8.2 Calculate $p_0(z)$ and $\rho_0(z)$.

First, we collect (8.1), (8.2), (8.6),

$$p_0 = \rho_0 R_d T_0, \tag{1}$$

$$\frac{dp_0}{dx_3} = -\rho_0 g, \tag{2}$$

$$\frac{dT_0}{dx_3} = -\frac{g}{c_p}. \tag{3}$$

The last, temperature equation can be solved immediately, based on the surface temperature. Density can be eliminated from the first two equations.

$$T_0 = T_s - \frac{g x_3}{c_p}, \tag{4}$$

$$p_0 = \frac{-1}{g}\frac{dp_0}{dx_3} R_d \left(T_s - \frac{g x_3}{c_p}\right). \tag{5}$$

The last equation can be simplified and integrated by collecting terms

$$\frac{-g}{R_d \left(T_s - \frac{g x_3}{c_p}\right)} = \frac{1}{p_0}\frac{p_0}{x_3}, \tag{6}$$

$$\frac{c_p \ln\left(\left[c_p R_d T_s - g R_d T_s\right] / c_p R_d T_s\right)}{R_d} = \ln\frac{p_0}{p_s}. \tag{7}$$

Thus,

$$T_0 = T_s - \frac{g x_3}{c_p}, \tag{8}$$

$$p_0 = p_s \left(1 - \frac{g x_3}{c_p T_s}\right)^{c_p/R_d}, \tag{9}$$

$$\rho_0 = \frac{p_s}{R_d T_s}\left(1 - \frac{g x_3}{c_p T_s}\right)^{c_p/R_d - 1} \tag{10}$$

NOTE: Wyngaard (2010) is accessible at `https://login.revproxy.brown.edu/login?url=http://search.ebscohost.com/login.aspx?direct=true&scope=site&db=e000xna&AN=324086`

2 Wyngaard (2010) Problem 8.6

8.6 Show that mixing ratio is a conserved variable.

I'm skipping all tildes... We show c is conserved by showing its conservation equation can be derived from that of ρ and ρ_c:

$$\frac{Dc}{Dt} = \frac{D}{Dt}\frac{\rho_c}{\rho} = \frac{1}{\rho}\frac{D\rho_c}{Dt} - \frac{\rho_c}{\rho^2}\frac{D\rho}{Dt} = \frac{1}{\rho}\underbrace{\left[\cancel{-\rho_c\frac{\partial u_i}{\partial x_i}} + \gamma\frac{\partial^2\rho_c}{\partial x_i\partial x_i}\right]}_{(8.31)} - \frac{\rho_c}{\rho^2}\underbrace{\left[\cancel{-\rho\frac{\partial u_i}{\partial x_i}}\right]}_{(8.10)}, \tag{11}$$

$$= \frac{\gamma}{\rho}\frac{\partial^2\rho_c}{\partial x_i\partial x_i} \approx \gamma\frac{\partial^2 c}{\partial x_i\partial x_i} \tag{12}$$

Where the last step assumes the variations in mixing ratio are much larger than the variations in either density.

3 Wyngaard (2010) Problem 9.6

9.6 Sketch the profile of vertical temperature flux in a quasi-steady convective ABL capped by an inversion.

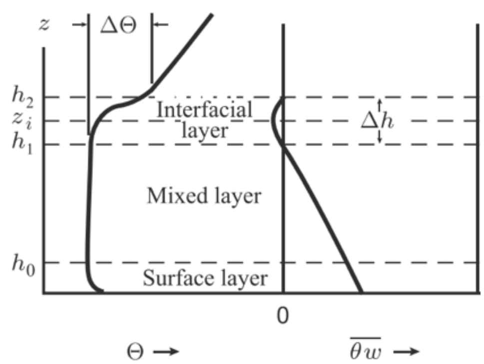

Figure 1: **Fig 11.1** excerpt

4 Wyngaard (2010) Problem 9.15

9.15 Write an expression for a turbulence Rossby number, the ratio of typical inertial and Coriolis forces on energy-containing eddies. Estimate its magnitude in the ABL.

The turbulence Rossby number Ro_* is the turbulent velocity scale divided by the Coriolis parameter and turbulence lengthscale: $u^*/f\ell$. This can be estimated a number of ways, here are some:

$$1 : \mathrm{Ro}_* = \frac{1m/s}{10^{-4}/s \;\; 10^3 m} = 10, \tag{13}$$

$$2 : 0.25 \approx \mathrm{Ri}_f = \frac{\frac{g}{\theta_0}\overline{\theta w}}{\overline{u_i u_j}\frac{\partial U}{\partial x_j}} \sim \frac{g d\theta}{\theta_0}\frac{\ell}{u_*^2} = \frac{g d\theta}{\theta_0 \ell}\frac{1}{f^2}\frac{f^2\ell^2}{u_*^2} \sim \frac{N^2}{f^2}\frac{1}{\mathrm{Ro}_*^2} \tag{14}$$

$$\therefore \mathrm{Ro}_* \approx \frac{1}{\sqrt{\mathrm{Ri}_f}}\frac{N}{f} \approx 2\frac{L_d}{\ell} \gg 2, \tag{15}$$

$$3 : \frac{\partial \overline{uw}}{\partial z} = f\left(V - V_g\right), \tag{16}$$

$$\therefore \mathrm{Ro}_* = \frac{u_*}{f\ell} = \sqrt{\frac{1}{f^2\ell}\frac{u_*^2}{\ell}} \sim \sqrt{\frac{1}{f^2\ell}\frac{\overline{uw}}{\partial z}} = \sqrt{\frac{1}{f\ell}\left(V - V_g\right)} \sim \sqrt{\frac{|V_g|}{f\ell}} = \sqrt{\mathrm{Ro}\frac{L_d}{\ell}} \gg 1 \tag{17}$$

Where Ro is the large scale flow Rossby number, which is typically less than one and thus $\sqrt{\mathrm{Ro}}$ is closer to one.

5 Wyngaard (2010) Problem 10.3

10.3 Derive an expression like Eq. (10.21) but for the surface temperature flux.

We take as inputs $Q_0, U_{ref}, z_{ref}, z_0, L, z_T, \Theta_{ref}, Q_{ref}$ where the z_T serves the same role as z_0 but in temperature, which might be different due to e.g., thermal over a variety of surfaces. In practice, it's probably normally acceptable to have $z_0 \approx z_T$. By definition, we have

$$Q_0 = -T^* u^* \tag{18}$$

There are 8 dimensional parameters, and 3 units, so we expect 5 dimensionless groups. Thus,

$$\frac{Q_0}{\Theta_{ref} U_{ref}} \equiv c_h = c_h\left(\frac{z_{ref}}{z_0}, \frac{z_{ref}}{z_0}, \frac{z_{ref}}{L}\right) \tag{19}$$

Following the example of (10.21), we can take $Q_{ref} \equiv \Theta_{ref} U_{ref}$ as the definition of Θ_{ref}, or we can use the mean temperature profile to do so instead of the flux, in which case we would need another dimensionless parameter to compare Q_{ref} to $\Theta_{ref} U_{ref}$, such as a Nusselt number. Note that in M-O similarity, it is argued that the mean velocity can be neglected to preserve Gallilean invariance (although it is hard to argue that this is relevant over a rough surface, which might have specific features that synch with the overlying turbulence.)

6 Wyngaard (2010) Problem 10.12

10.12 Explain the role of the second-moment budgets in turbulent flow calculation.

Second moment budgets serve a variety of roles.
The first, which is exemplified by many of the examples in Chapter 10, is to first examine the mean flow equations and which second moments likely contribute, then use the second moment budgets to estimate the behavior of these terms. This approach could be said to be scaling or parameterizing the second moments.
A second approach is a *second moment closure* approach. In this approach (e.g. ?), all of the mean profile equations and second moment budgets are written together as PDEs in z and t. As they share many terms in common (e.g., the buoyancy production term in the diagonal Reynolds stresses is also the vertical transport of buoyancy in the buoyancy equation and part of the turbulent transport of buoyancy variance), they form a coupled set of equations for more unknowns than there are equations (the turbulent closure problem). By neglecting or parameterizing just enough of these unknowns, a closed set of equations can be formed. Finally, by expanding about a steady solution or a set of basis functions, these equations are converted into an algebraic set of equations instead of differential equations.
A third and final approach is used when observations or Large Eddy Simulations allow direct calculation of the second moments. In this case, they can be used either in the mean equations to understand the coupling of the turbulence and the mean or they can be used together to attempt to close the second moment budgets, with any residuals in failing to close the budgets serving as an estimate of the terms that were unable to be measured (e.g., the triple product terms, which are noisier and harder to measure).

7 Wyngaard (2010) Problem 11.2

11.2 Develop a criterion for the negligibility of the effects of horizontal inhomogeneity and time changes on mixed-layer similarity.

These constraints can really be found whenever a dominant balance is thought to be in jeopardy of being violated. Let's take the three similarity equations (10.12) as a starting point,

$$\frac{kz}{u_*}\frac{\partial U}{\partial z} = \phi_m\left(\frac{z}{L}\right), \quad -\frac{kzu_*}{Q_0}\frac{\partial \Theta}{\partial z} = \frac{kz}{T_*}\frac{\partial \Theta}{\partial z} = \phi_h\left(\frac{z}{L}\right),$$
$$-\frac{kzu_*}{C_0}\frac{\partial C}{\partial z} = \frac{kz}{c_*}\frac{\partial C}{\partial z} = \phi_c\left(\frac{z}{L}\right), \qquad (10.12)$$

If we examine the mean momentum equations,

$$\frac{\partial \overline{uw}}{\partial z} = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + fV, \qquad \frac{\partial \overline{vw}}{\partial z} = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} - fU. \qquad (11.3)$$

we see that the assumptions of homogeneity and stationarity imply retaining a turbulent term which must be larger than the time derivative or the advective term,

$$\frac{1}{T}\frac{u_*}{k} \sim \left|\frac{\partial U}{\partial t}\right| \ll \left|\frac{\partial \overline{uw}}{\partial z}\right| \sim \frac{u_*^2}{z} \tag{20}$$

$$1 \gg \frac{z}{u_* T} \approx \frac{|L|}{u_* T} \tag{21}$$

That is, the time scale of variations should be slow enough that it is not relevant at the M-O height. For spatial scales, the same procedure follows, except one compares the advection of divergence of stresses to the shear production term. Thus,

$$U\frac{u_*^2}{L_x} \ll \frac{u_*^3}{z} \approx \frac{u_*^3}{|L|}, \tag{22}$$

$$\frac{U}{u_*}|L| \ll L_x \tag{23}$$

So, the horizontal must be long compared to the M-O depth, scaled by the ratio of the mean velocity to the friction velocity. The latter ratio does not appear if it is only the *turbulence* that is heterogeneous, not the mean flow, which can occur if there are variations in roughness. In that case, the lengthscale of roughness variation must obey $|L| \ll L_x$.

8 Wyngaard (2010) Problem 12.2

> 12.2 A surface is cooler than the air above and is evaporating water so that the vertical flux of water vapor is positive. The virtual temperature flux is zero. What is the stability index z/L? Using M-O similarity, write the expression for the vertical gradient of potential temperature.

Here we realize that Q_0 scales for the *buoyancy*, not the temperature, as it appears as a scaling in the potential energy equation not the thermal energy equation. Thus, we consider the Q_0 to be the surface flux of virtual temperature. Consulting equation (10.11),

$$\tilde{\theta}_v = \tilde{T}(1 + 0.61\tilde{q}) \tag{24}$$

$$\overline{w\theta_v} = Q_0 \propto T_* = 0 \tag{25}$$

$$\overline{w\theta} = -0.61T\overline{wq} \tag{26}$$

$$\lim_{Q_0 \to 0} \frac{z}{L} = 0 \tag{27}$$

Thus, the surface layer rules apply, where we expect a nearly constant flux of temperature and

$$\frac{\partial \theta_v}{\partial z} = \frac{T_*}{kz} = 0 \tag{28}$$

$$\frac{\partial \theta_v}{\partial z} = \frac{-\overline{w\theta}}{u_* kz}\phi_\theta\left(\frac{z}{L}\right) = \frac{0.61T\overline{wq}}{u_* kz}\phi_\theta\left(\frac{z}{L}\right). \tag{29}$$

The surface layer scalings, furthermore, provide the limiting value of $\phi(0) \approx 1$, so

$$\frac{\partial \theta_v}{\partial z} \approx \frac{-\overline{w\theta}}{u_* kz} = \frac{0.61T\overline{wq}}{u_* kz}. \tag{30}$$

References

Wyngaard, J. C. (2010). *Turbulence in the Atmosphere*. Cambridge University Press.