The Explicit, Stochastic, Coordinate-independent (ESC) Framework Project

Baylor Fox-Kemper & Joel Feske (Brown U.),
Jonathan Lilly (Planetary Science Institute)

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Overview

Overview: To advance the development and assessment of parameterizations of eddy transport in numerical models, the creation and application of a new theoretical framework is proposed. The framework is not a new parameterization, but rather a set of principles linking physical meaning to parameterization form, including extant as well as yet-unformulated variations. The result will be a gain in perspective that will support and spur parameterization progress and evolution.

ESC: Explicit

- (i) explicit and rigorous definitions for filtering and averaging operations, clearly delineating what is meant by "mean";

Following Sadek and Aluie (2018) and Salmon (2013), we define a generalized average tracer concentration—including time-, space-, and ensemble-averaging—as

$$\overline{\tau}(\mathbf{x},t) = \int \mathrm{d}t' \iiint \mathrm{d}^3\mathbf{x}' \int \mathrm{d}\mu G(\mathbf{x}-\mathbf{x}',t-t',\mu)\tau(\mathbf{x}',t',\mu) \quad (2)$$

where $\mathbf{x}'$ and t' are dummy space and time variables, each member of the ensemble is assigned a unique value of the ensemble parameter μ, and $G(\mathbf{x},t,\mu)$ is a normalized convolution kernel

We note that isosurfaces of G need not be isotropic nor follow Cartesian directions

–indeed it is often useful for other choices to be made.

Coarse vs. Fine S. Ocean Eddies

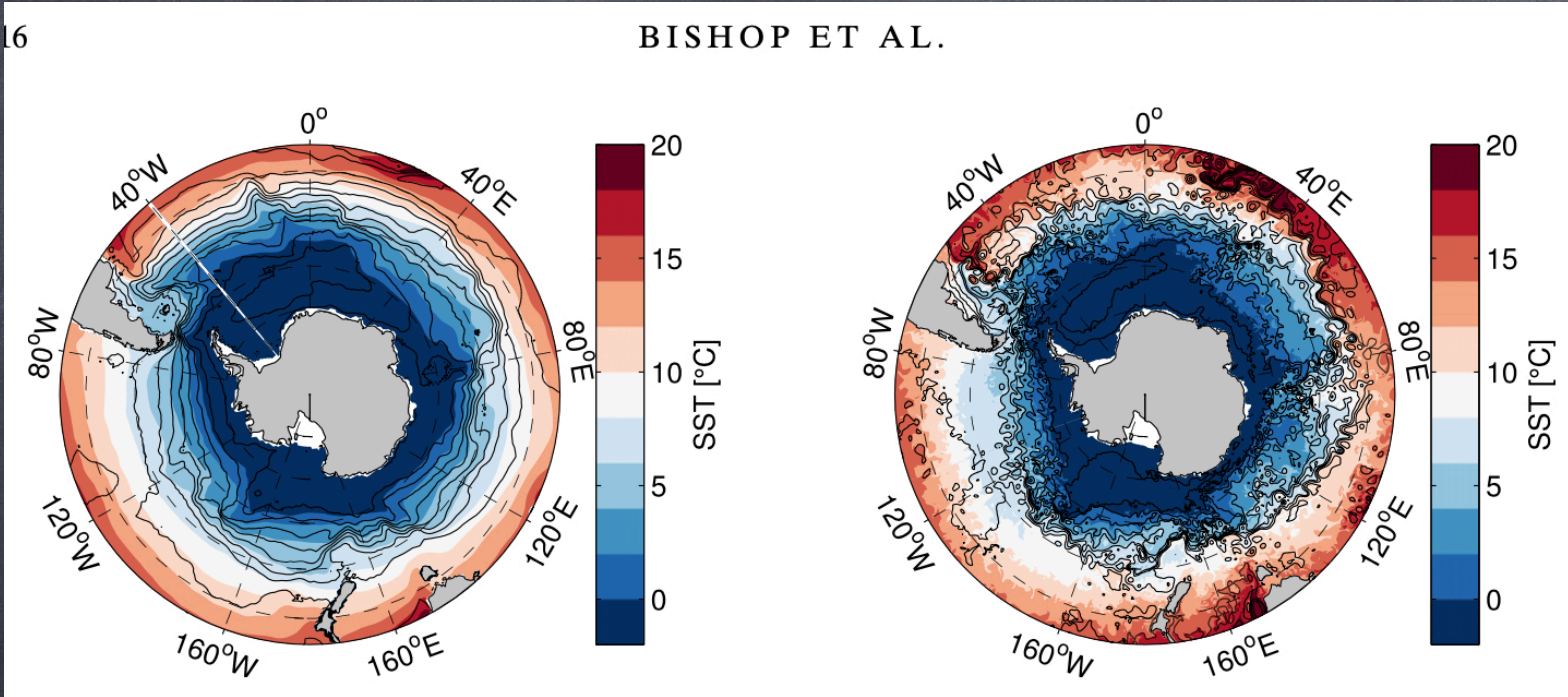
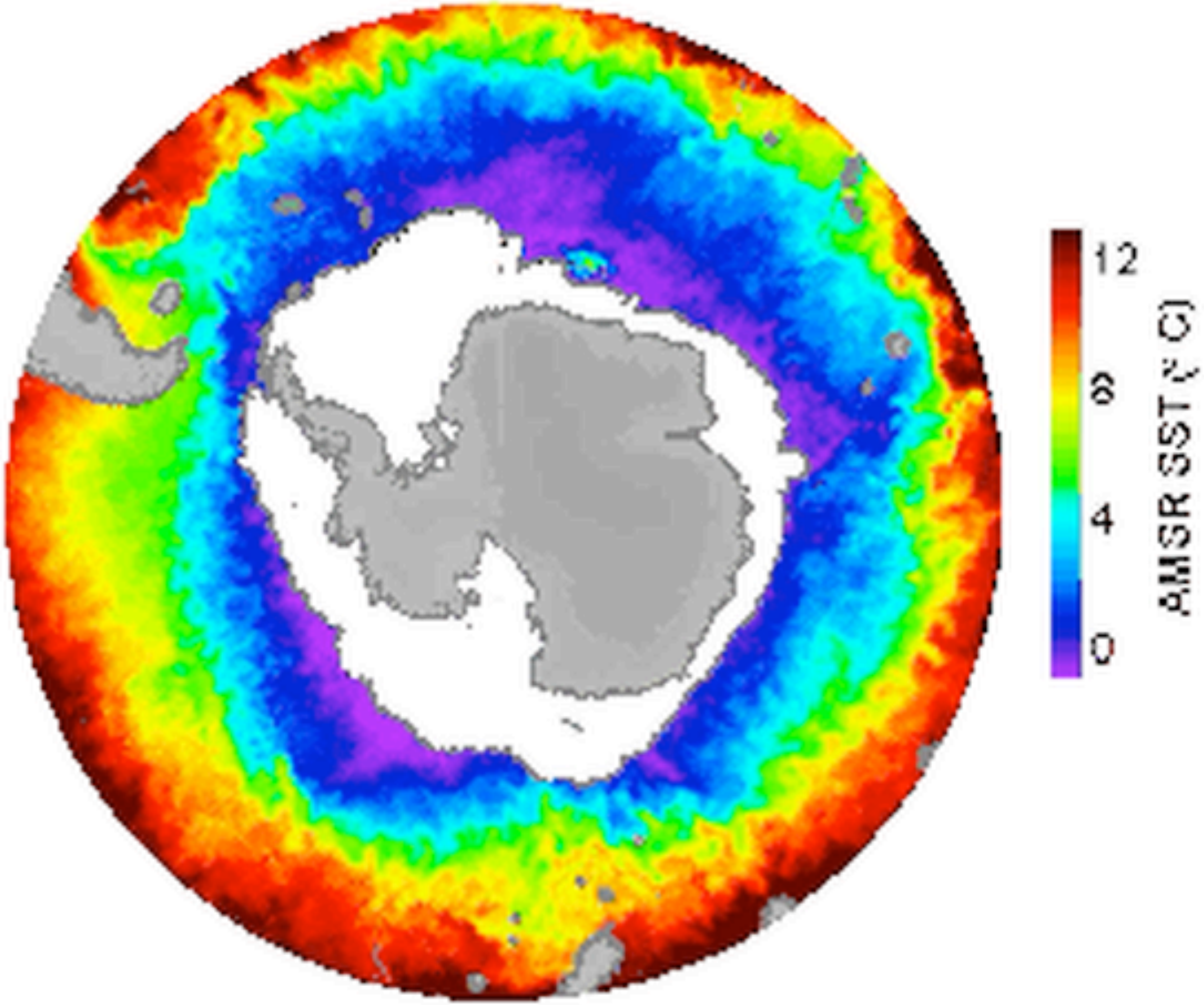


FIG. 1. Snapshots of austral winter SST and SSH in (left) low- (1°) and (right) high-resolution (0.1°) CESM. Color contours are SST [contour interval (ci) = 2°C] and black contours are SSH (ci = 25 cm).

Bishop, S.P., Gent, P.R., Bryan, F.O., Thompson, A.F., Long, M.C. and Abernathey, R., 2016. Southern Ocean overturning compensation in an eddy-resolving climate simulation. Journal of physical oceanography, 46(5), pp.1575-1592.

AMSR SST Date: 06/03/2002



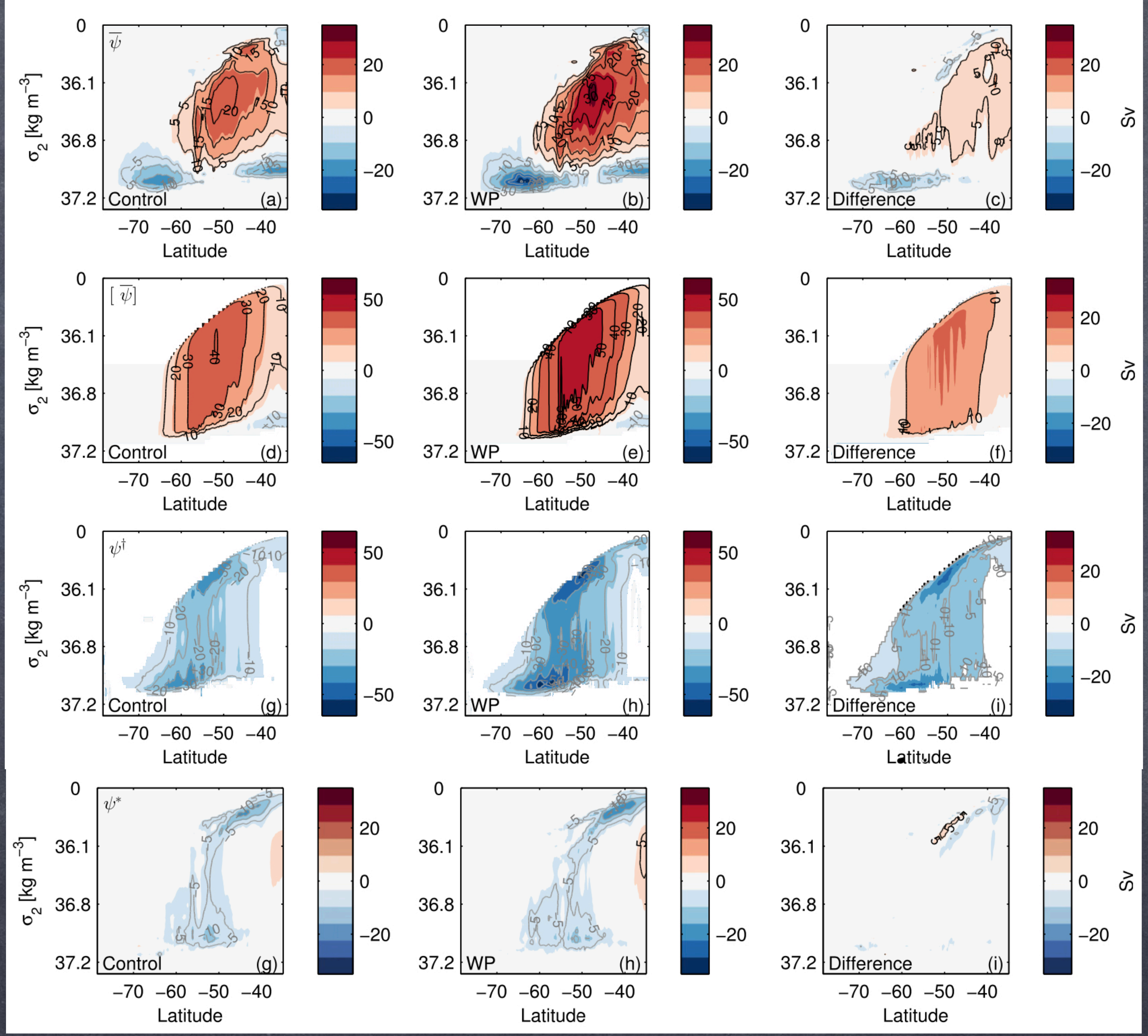
Standing vs. Transient Eddies

Time mean

Time & zonal mean

Standing eddy

Transient eddy



Often these two are parameterized together!

Bishop, S.P., Gent, P.R., Bryan, F.O., Thompson, A.F., Long, M.C. and Abernathey, R., 2016. Southern Ocean overturning compensation in an eddy-resolving climate simulation. Journal of physical oceanography, 46(5), pp.1575-1592.

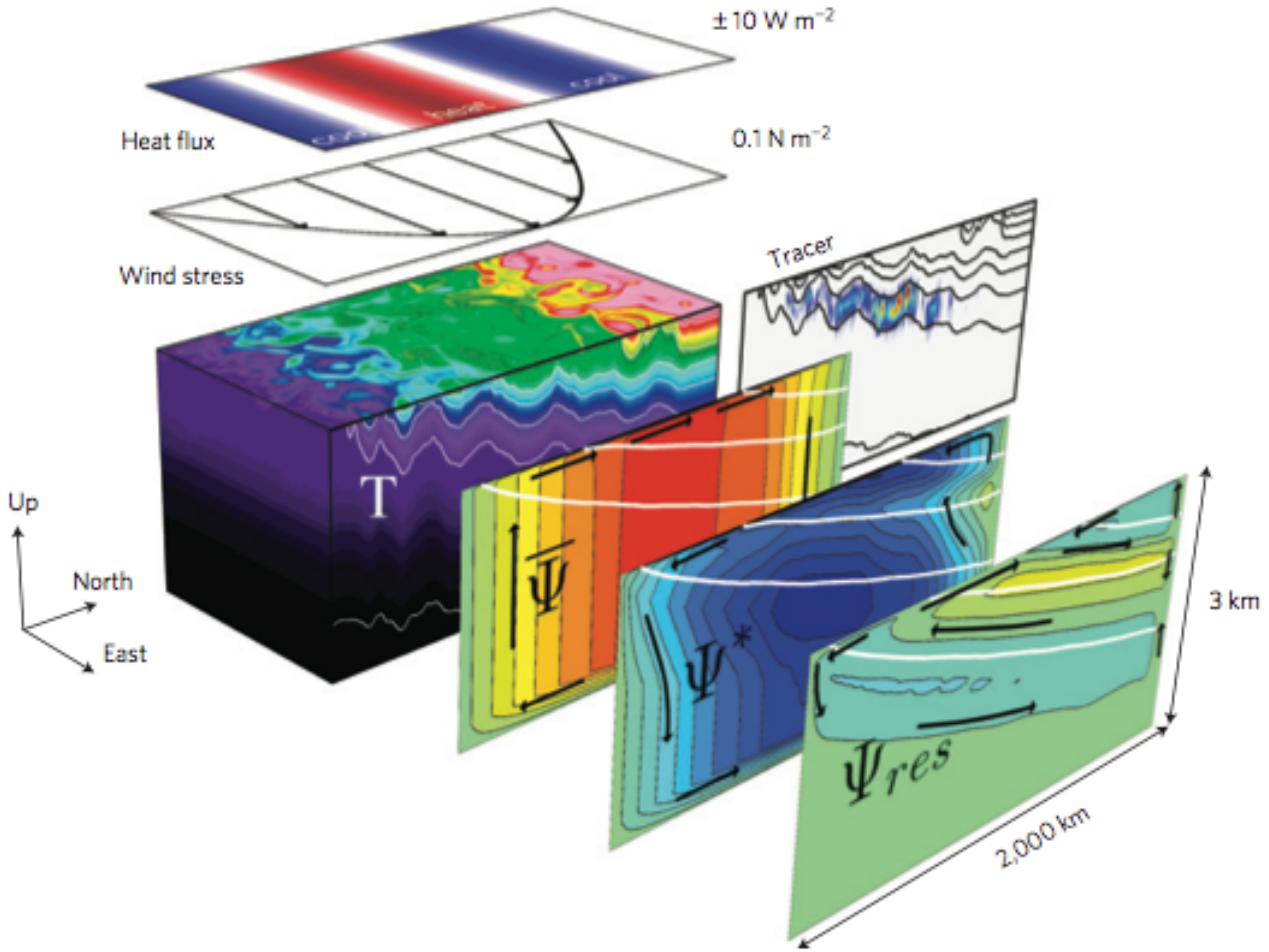


Figure 5 | Idealized simulations of the ACC. Results from numerical simulations of an eddying re-entrant channel showing the wind and surface heat fluxes driving the channel flow (overlaid above); an instantaneous three-dimensional snapshot of the model's temperature field T (here coincident with density $\rho = \rho(T)$), with two density surfaces picked out in white, undulating in concert with the mesoscale eddy field; and time-mean overturning cells $\overline{\psi}$, ψ^* and ψ_{res} (computed as defined in equation (1)) with time-mean density surfaces plotted in white. Also shown is an instantaneous section of tracer released into the flow. Antarctica is imagined to be on the left. The model is the MITgcm run at a horizontal resolution of 4 km over a 1,000 km by 3,000 km domain. The cooling (blue), warming (red), cooling pattern of air-sea fluxes, on moving out from Antarctica, are arranged to be reminiscent of Fig. 4b, and lead to the particular pattern of upper and lower residual overturning cells seen in ψ_{res}. More details can be found from Abernathey and colleagues[45].

Depth-Integrated Energy Budgets: Layered vs. Thickness-Weighted

$$\mathrm{MKE} = \frac{1}{2}\sum_{n=1}^{N} \bar{h}_n |\overline{\boldsymbol{u}}_n|^2,$$

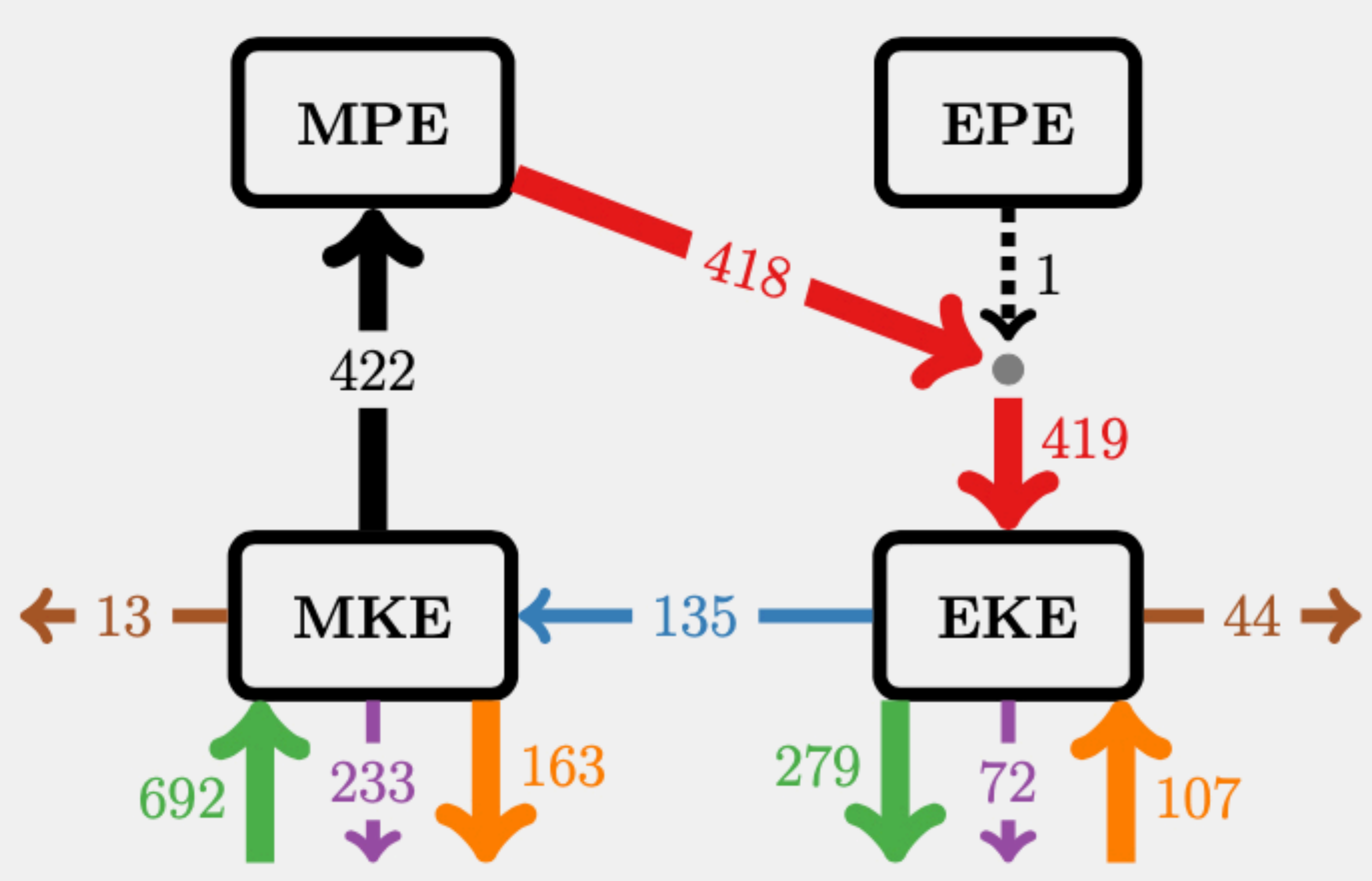
$$\mathrm{EKE} = \overline{\mathrm{KE}} - \frac{1}{2}\sum_{n=1}^{N} \bar{h}_n |\overline{\boldsymbol{u}}_n|^2,$$

$$\hat{\boldsymbol{u}}_n = \frac{\overline{h_n \boldsymbol{u}_n}}{\bar{h}_n};$$

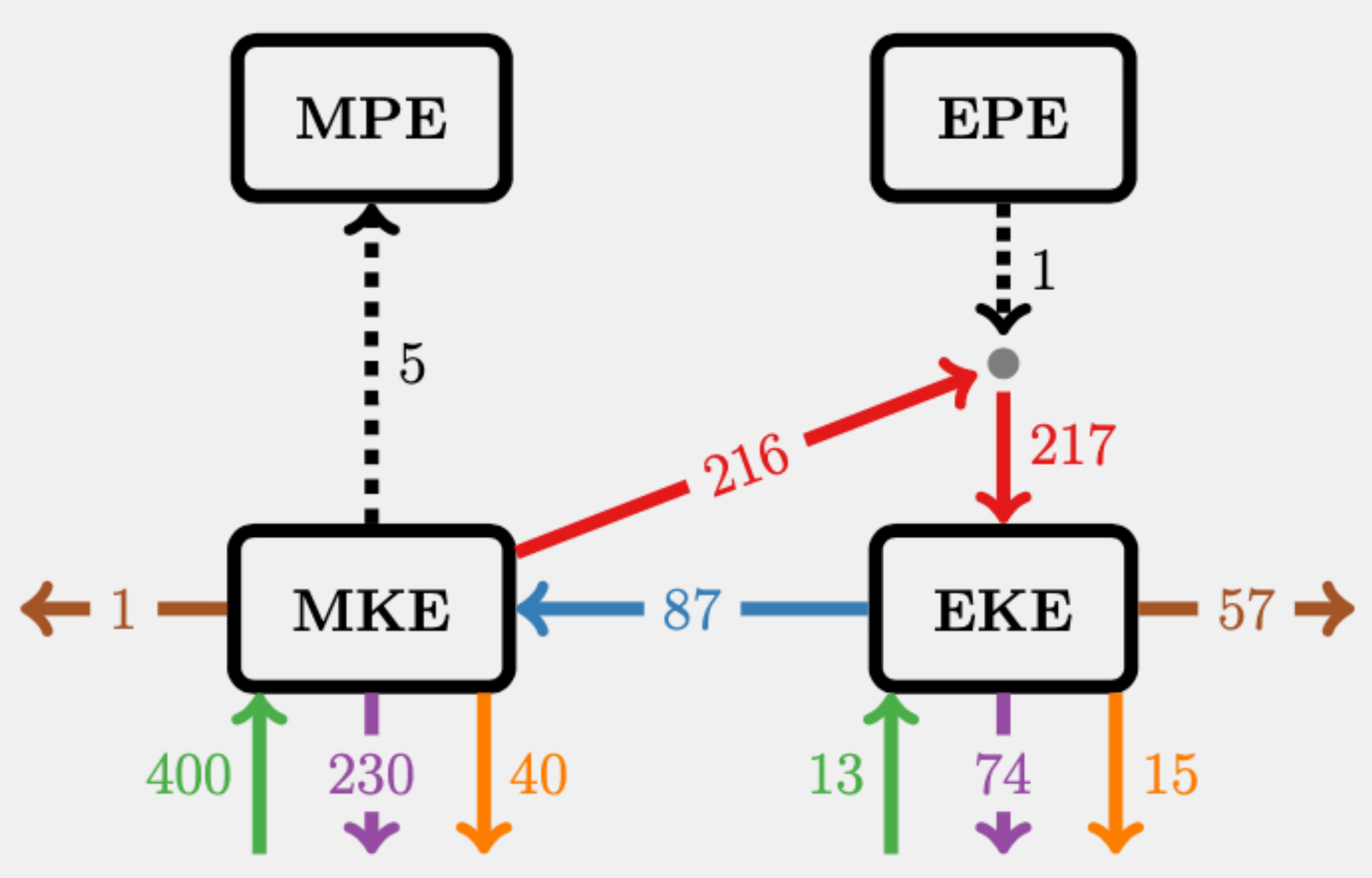
$$\mathrm{MKE} = \frac{1}{2}\sum_{n=1}^{N} \bar{h}_n |\hat{\boldsymbol{u}}_n|^2,$$

$$\mathrm{EKE} = \overline{\mathrm{KE}}_n - \frac{1}{2}\sum_{n=1}^{N} \bar{h}_n |\hat{\boldsymbol{u}}_n|^2.$$

(a) Lorenz-like energy cycle, filter scale: 1°



(c) Bleck energy cycle, filter scale: 1°



Diagnosing scale-dependent energy cycles in a high-resolution isopycnal ocean model

Nora Loose,[a] Scott Bachman,[b] Ian Grooms,[a] Malte Jansen[c]

ESC: Stochastic

- ii) a flexible approach to stochastic modeling that relaxes the unrealistic constraints implicit in existing parameterizations on eddy path smoothness and dependence on local conditions

This is of course the primary research topic of STUOD.
We look forward to learning from you & thinking about implementations of your insights.

ESC: Stochastic

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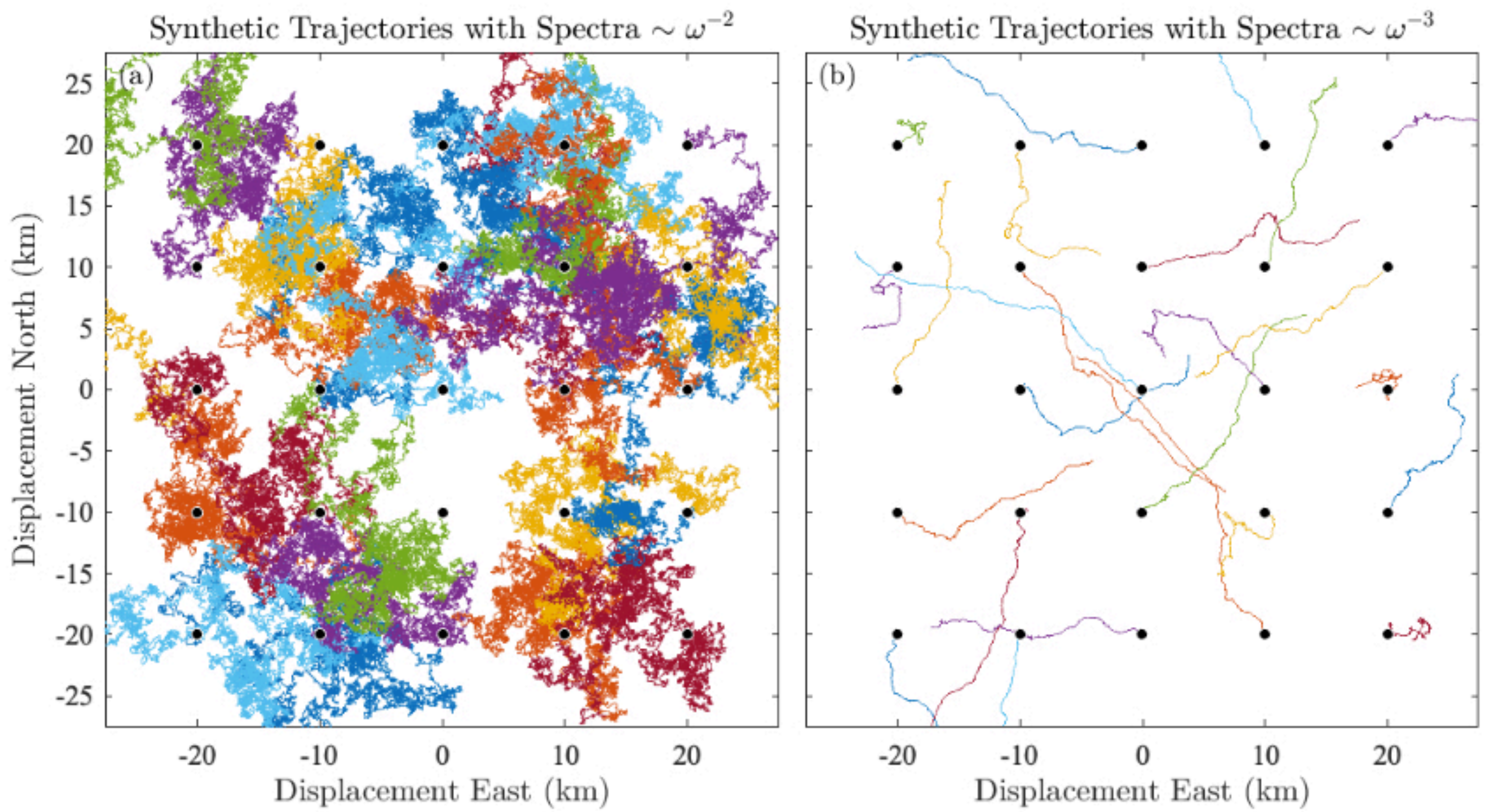
One test: Weiner vs. fractional Brownian vs. Matérn Eddies

Many eddy "diffusivities" (resulting in deterministic, diffusive PDEs for tracers) are constructed from Fokker-Planck equations of Weiner processes, as in the long timescale limit studied by Taylor (1921).

However, observations of ocean eddies reveal another story...



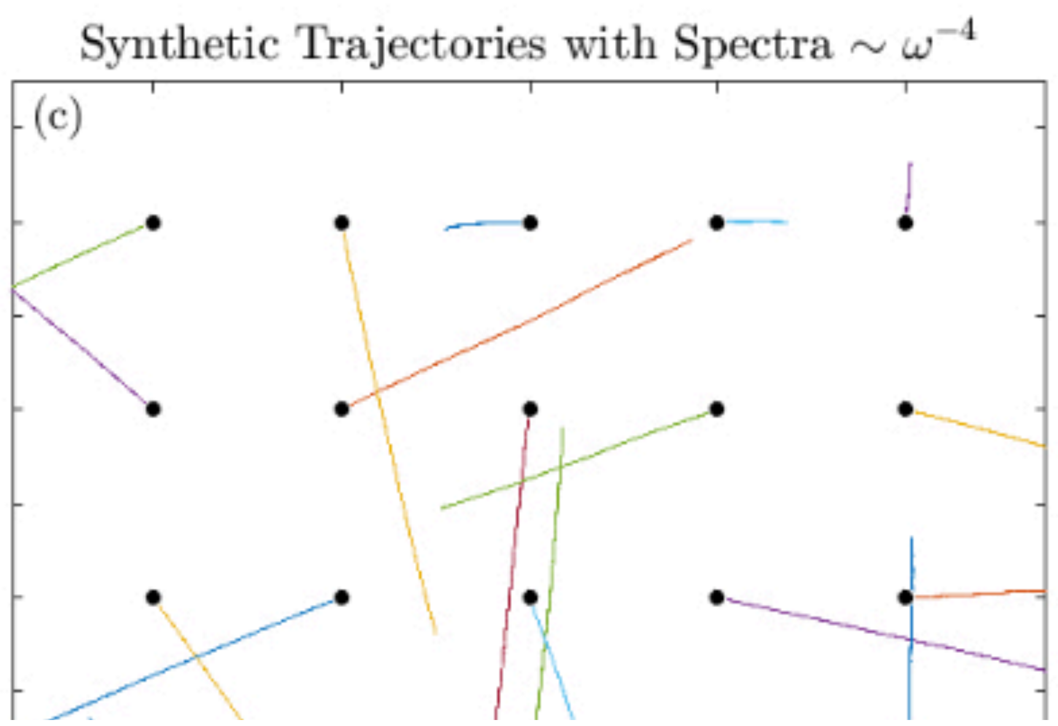
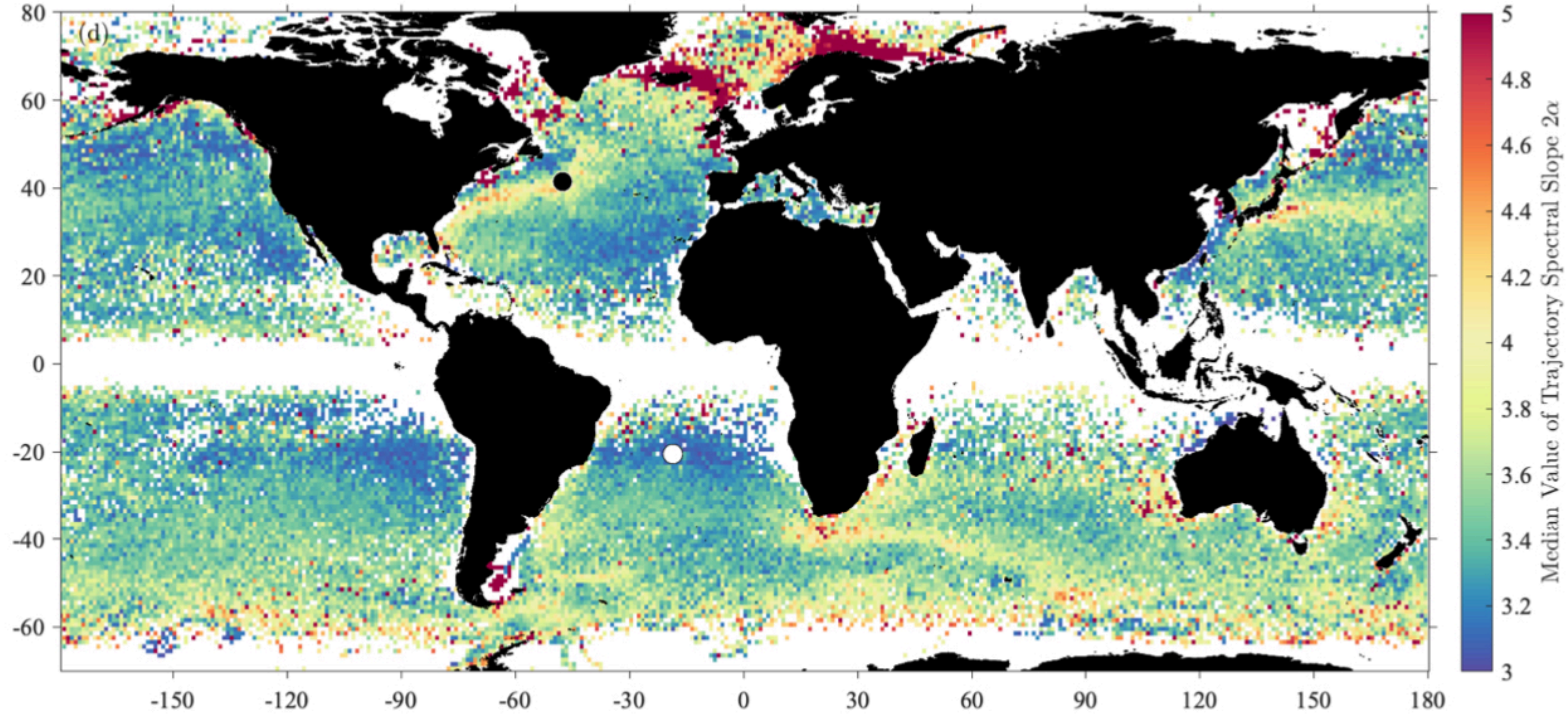


Figure 2: Synthetic Lagrangian trajectories consisting of fractional Brownian motion with spectra proportional to (a) ω^{-2}, (b) ω^{-3}, and (c) ω^{-4}, all set to unit variance, together with (d) estimated high-frequency spectral slopes for Lagrangian trajectories from NOAA's global drifter dataset. In panels (a), (b), and (c), dots show the launch points of the trajectories, while in (d), a white and a black dot mark locations with ω^{-3} and ω^{-4} spectra, as in panels (b) and (c). Trajectories with spectra behaving as ω^{-2} as shown in (a), corresponding to Brownian motion, are nowhere observed. From Lilly, Sykulski, and Early (2022), in prep.



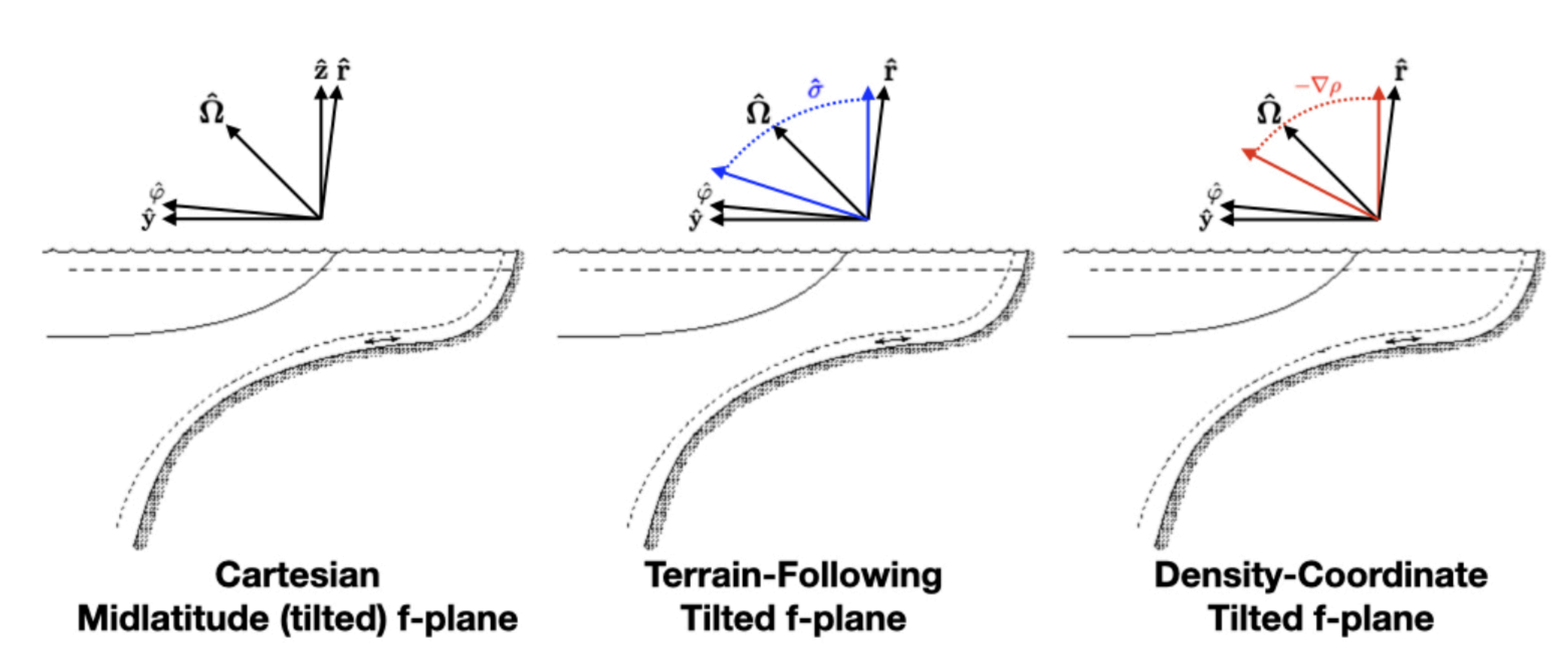
$$\sigma^2 = R_{zz}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{zz}(\omega)\,\mathrm{d}\omega,$$

$$\kappa = \frac{1}{4}S_{zz}(0) = \frac{1}{4}\int_{-\infty}^{\infty} R_{zz}(\tau)\,\mathrm{d}\tau,$$

$$S_{zz}(\omega) = \frac{A^2}{(\omega^2+\lambda^2)^{\alpha}},$$

ESC: Coordinate-independent

- (iii) the use of coordinate-independent representations to express the physical essence of parameterizations free from any particular choice of model geometry, together with formal rules for realizing each formulation in any given coordinate system



ESC: Coordinate-independent

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An Example:

$$-\left(\overline{u^i\tau} - \overline{u^i}\;\overline{\tau}\right) \approx \mathbb{K}^{ij}\partial_j\overline{\tau} = \left(\mathbb{A}^{ij} + \mathbb{S}^{ij}\right)\partial_j\overline{\tau}$$

The required expressions for $\mathbb{A}^{ij}$ and $\mathbb{S}^{ij}$ do not appear in the literature. However, in preliminary work (Feske and Fox-Kemper, 2022), we find as a candidate expression for the Redi tensor

A transverse-isotropic diffusivity

$$\mathbb{S}^{ij} = \kappa\,\frac{g^{ij}\,\partial^k\rho\,\partial_k\rho - \partial^i\rho\,\partial^j\rho}{\partial^k\rho\,\partial_k\rho} \qquad (5)$$

In Cartesian coords, Equals Redi form

where κ is the diffusivity (assumed constant for simplicity herein), g^{ij} is the metric tensor, and $\partial_i \equiv \partial/\partial q^i$ and $\partial^i \equiv \partial/\partial q_i$ are covariant and contravariant spatial derivatives respectively. The significance of Eqns. (4)-(5) is their generality: they apply *in any smooth, differentiable coordinate system* and with *any definition of "mean" that is expressible via the convolution operator in Eqn. (2).*

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An Example:

find the Redi tensor in any chosen coordinate system. For example, in isopycnal coordinates with constant κ, and with subscripts indicating partial derivatives, we have

$$\mathbf{S} = \frac{\kappa}{h_x^2 + h_y^2 + h_z^2} \begin{bmatrix} h_y^2 + h_z^2 & -h_x h_y & -h_x h_\rho \\ -h_x h_y & h_x^2 + h_z^2 & -h_y h_\rho \\ -h_x h_\rho & -h_y h_\rho & h_\rho^2 \frac{h_x^2 + h_y^2}{h_z^2} \end{bmatrix} \approx \frac{\kappa}{h_x^2 + h_y^2 + 1} \begin{bmatrix} h_y^2 + 1 & -h_x h_y & 0 \\ -h_x h_y & h_x^2 + 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6}$$

where $h \equiv z_\rho$ is the isopycnal layer thickness. The first expression emerges from simply inserting the correct metric tensor into Eqn. (5). The latter approximation follows if h_ρ can be neglected, meaning that the stratification is uniform in the vertical; this reproduces a form presented by Gent and McWilliams (1990), see their Eqn. (4). The former equation, the Redi tensor in isopycnal coordinates if the assumption of uniform stratification is not made, is apparently a new result.

Such corrections become increasingly important with higher-order numerical discretization

ESC: Coordinate-independent

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We can apply the coordinate-independent forms of fluid dynamics developed for astrophysics and cosmology. However, to arrive at the equations meaningful for use in oceanography, we need:

1) Rotating coordinates, not inertial frame
2) Easy application of hydrostatic, Boussinesq, other approx.
3) Differential geometry interpretations that are easy to use with complex shapes, e.g., density & pressure coordinates
4) Automatic evaluation of difficult quantities, e.g., Christoffel symbols (xAct)

The proposed work will create a new, comprehensive foundation for understanding eddy-driven transport and its parameterization, the **ESC** framework, which will have the following properties:

Explicit. The definition of “mean” will be given an explicit mathematical definition, drawing on recent filtering research, to encompass standard Reynolds averaging as well as more general types of averages. Far from being trivial, this essential step avoids ambiguities and misunderstandings, and enables the realization of the subsequent two properties. **(Section 3.1)**

Stochastic. At the core of the framework will be a flexible and inclusive stochastic model for the behavior of unresolved Lagrangian motion. By deriving and analyzing new classes of Fokker-Planck equations for the evolving probabilities associated with stochastic models of Lagrangian trajectories, the impact of trajectory smoothness will be assessed and incorporated constructively into parameterizations for the first time. **(Section 3.3)**

Coordinate-independent. The power of differential geometry will be exploited to compare parameterizations on a uniform footing, to reveal which statements are rigorously equivalent in any coordinate system and which are coordinate-dependent, and to allow us to readily express parameterizations into all major extant and nascent coordinate systems, including non-orthogonal, curvilinear, and Lagrangian coordinates. **(Section 3.2)**

The net result will be a theoretical framework within which assessments can be made, misunderstandings clarified, and inadequacies identified from first principles, a capability for which there is a clear and timely need and which is being addressed by no existing effort.

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