The Geometry of Advection, Diffusion, and Viscosity

Baylor Fox-Kemper

Brown University

Reflecting collaborations with Scott Bachman, Frank Bryan, Scott Reckinger, Brodie Pearson, Stanley Deser

(Clara's Dad Who taught me about diff. geometry and gravitational waves before LIGO!)

AGU Ocean Sciences: Tuesday, February 23, 2016 15:45 PM - 16:00 208-209

Supported by NSF 1350795

1D: Taylor '21 Einstein '05

- Einstein shows statistics of concentration of discrete particles, combined Brownian velocities behave like (Lagrangian) diffusion:

$$c_t(x,t) \approx D c_{xx}(x,t)\,, \qquad D \equiv \frac{\overline{\Delta^2}}{2\tau}.$$

- Taylor showed that the same is true of continuous movements, so long as they become decorrelated in time:

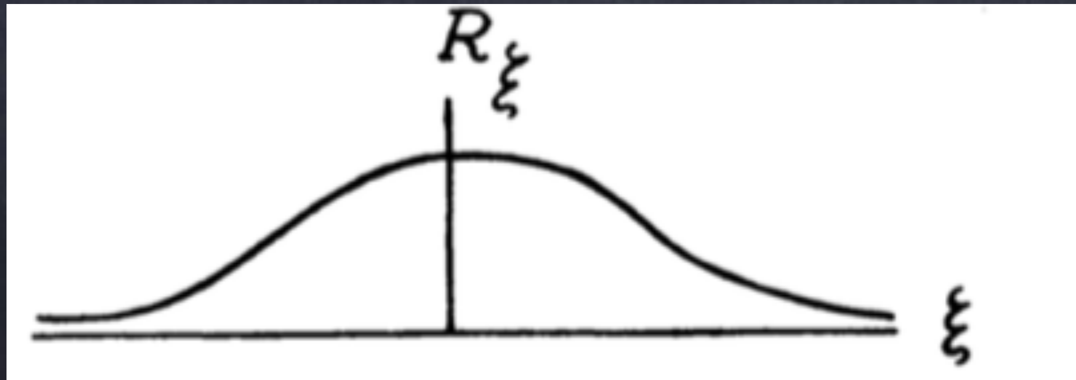


Fig. 2.

Velocity correlation

$$R_\xi = \overline{V(t)V(t+\xi)}$$

versus lag ξ

Thickness Weighted Mean and Favre-Average...

Lagrangian Diffusion implies different Lagrangian Advection

$$\mathbf{v} = \tilde{\mathbf{u}} + \boldsymbol{\nabla}_\rho \cdot \mathbf{K}.$$

Dukowicz & Smith '97

$$\mathbf{v} = \tilde{\mathbf{u}} + \boldsymbol{\nabla}_\rho \cdot \mathbf{K} + \frac{\mathbf{K} \cdot \boldsymbol{\nabla}_\rho \tilde{h}}{\tilde{h}} + \frac{\mathbf{K} \cdot \boldsymbol{\nabla}_\rho \hat{q}}{\hat{q}} + \frac{\widetilde{\zeta' \mathbf{u}'}}{\tilde{h}\hat{q}} + \mathrm{gt},$$

Dukowicz & Greatbatch '98

See also Young (2012)

tilde represents an average along an isopycnal surface

1D: Gedanken...

- Averaging can convert stochastic advection into diffusion. In the real & modeled world, we have only averaged fluxes and averaged tracer statistics.

- Can we tell Lagrangian advection from diffusion?

$$\frac{\partial}{\partial t}(\rho\tau) + \frac{\partial}{\partial x}\left[\underbrace{\rho u \tau}_{\text{advective}} \underbrace{- \rho\kappa\frac{\partial}{\partial x}\tau}_{\text{diffusive}}\right] = 0$$

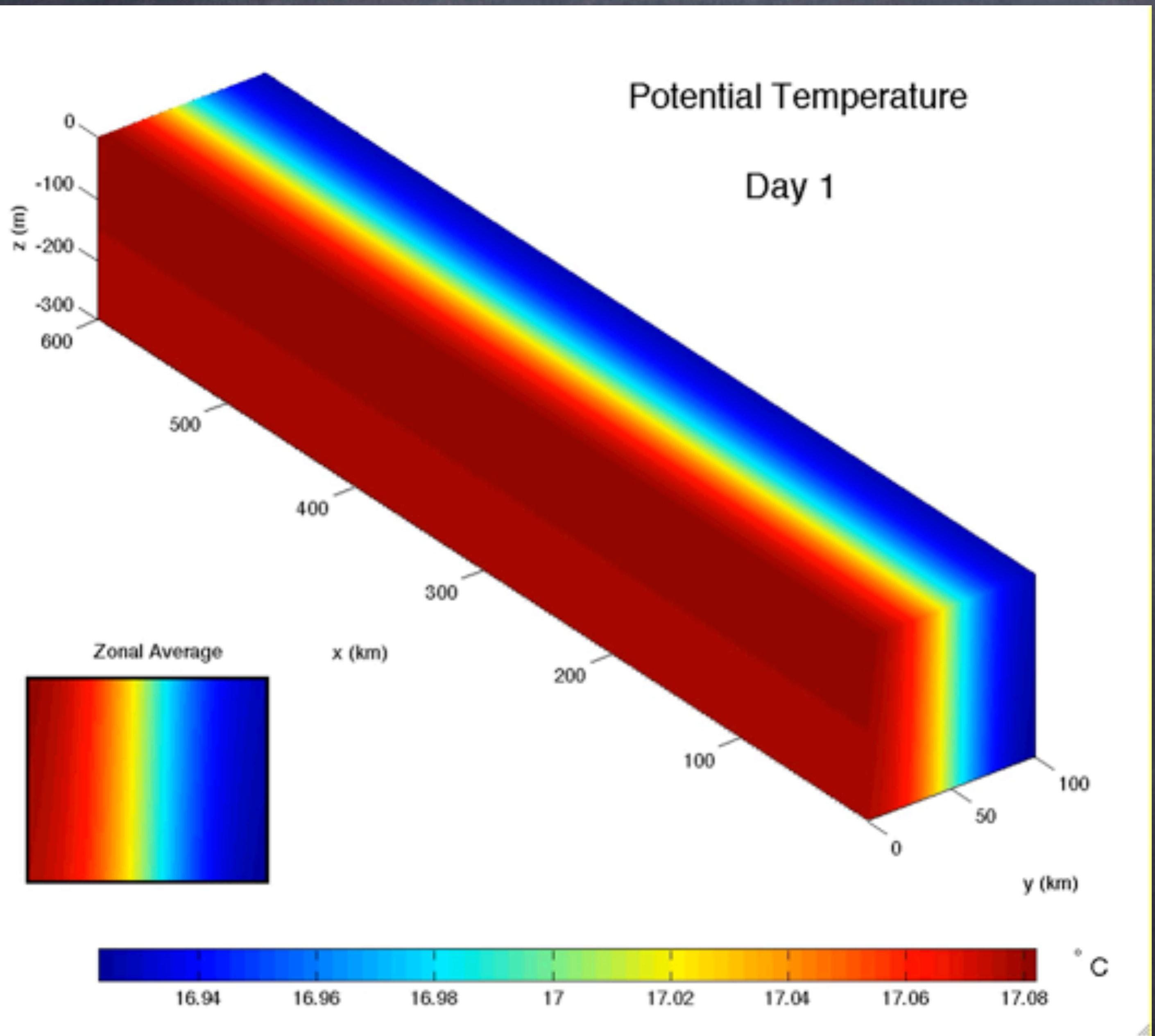
Well, you use more than one tracer, then just separate the flux into the part that's proportional to τ and the part that's proportional to $\frac{\partial \tau}{\partial x}$

- Note: all quantities here are some sort of average...

3d: Gedanken Donuts...

- In 3d: Can we tell advection from diffusion?

$$\partial_t(\rho\tau) + \partial_i \left[\underbrace{\rho v^i \tau}_{\text{advective}} - \underbrace{\rho \kappa^{ij} \partial_j \tau}_{\text{diffusive}} \right] = 0$$



- Note: all quantities here are some sort of average...
- Gauge uncertainty
- Which tracer?

S. Bachman, BFK, and F. O. Bryan. A tracer-based inversion method for diagnosing eddy-induced diffusivity and advection. Ocean Modelling, 86:1-14, February 2015.

Mesoscale Eddy Parameterizations all have the form:

$$\overline{\mathbf{u}'\tau'} = -\mathbf{R} \cdot \nabla \overline{\tau}$$

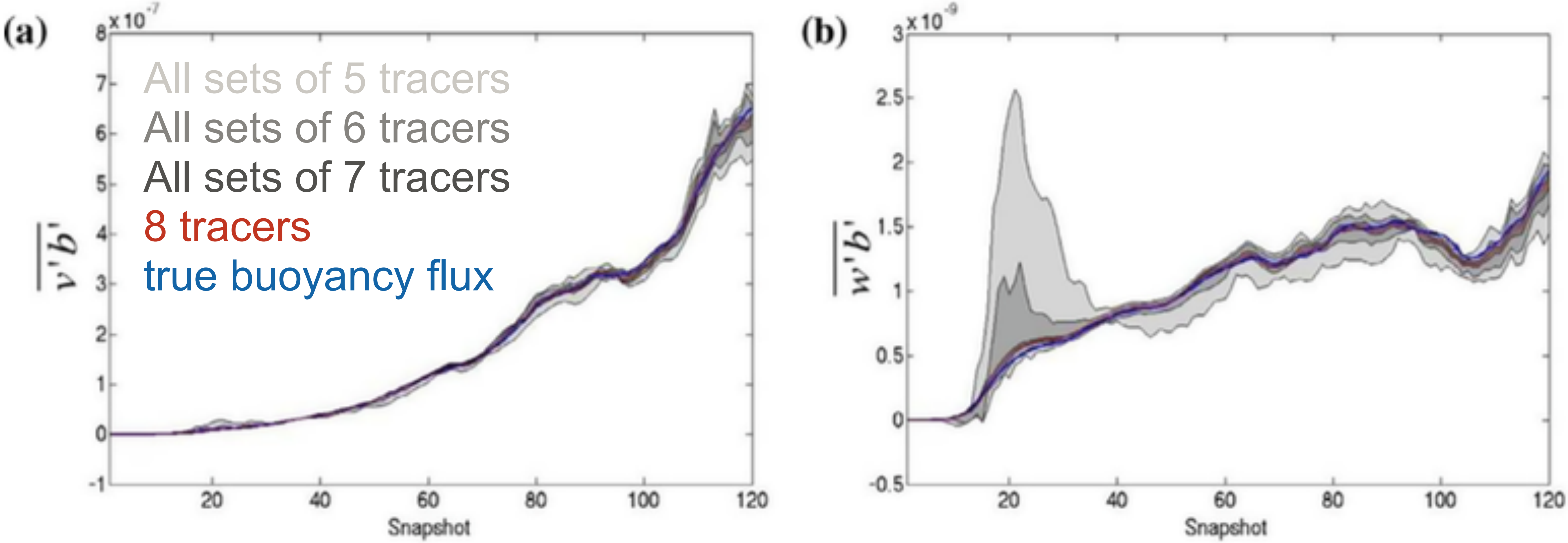
$$\begin{bmatrix} \overline{u'\tau'} \\ \overline{v'\tau'} \\ \overline{w'\tau'} \end{bmatrix} = - \begin{bmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{bmatrix} \begin{bmatrix} \partial_x \overline{\tau} \\ \partial_y \overline{\tau} \\ \partial_z \overline{\tau} \end{bmatrix}$$

- In Cartesian Coordinates (for the moment)
- Underdetermined, unless you use MULTIPLE TRACERS

BFK, R. Lumpkin, and F. O. Bryan. Lateral transport in the ocean interior. In G. Siedler, S. M. Griffies, J. Gould, and J. A. Church, editors, Ocean Circulation and Climate: A 21st century perspective, volume 103 of International Geophysics Series, chapter 8, pages 185-209. Academic Press (Elsevier Online), 2013.

With Enough Passive Tracers determining R, other tracers (e.g. buoyancy, PV) fluxes can be reconstructed.

R is approximately independent of tracer



$$\overline{\mathbf{u}'\tau'} = -\mathbf{R} \cdot \nabla\overline{\tau}$$

S. Bachman, BFK, and F. O. Bryan. A tracer-based inversion method for diagnosing eddy-induced diffusivity and advection. Ocean Modelling, 86:1-14, February 2015.

9 Tracers realistic high-res ocean; Drifters & high-res consistent

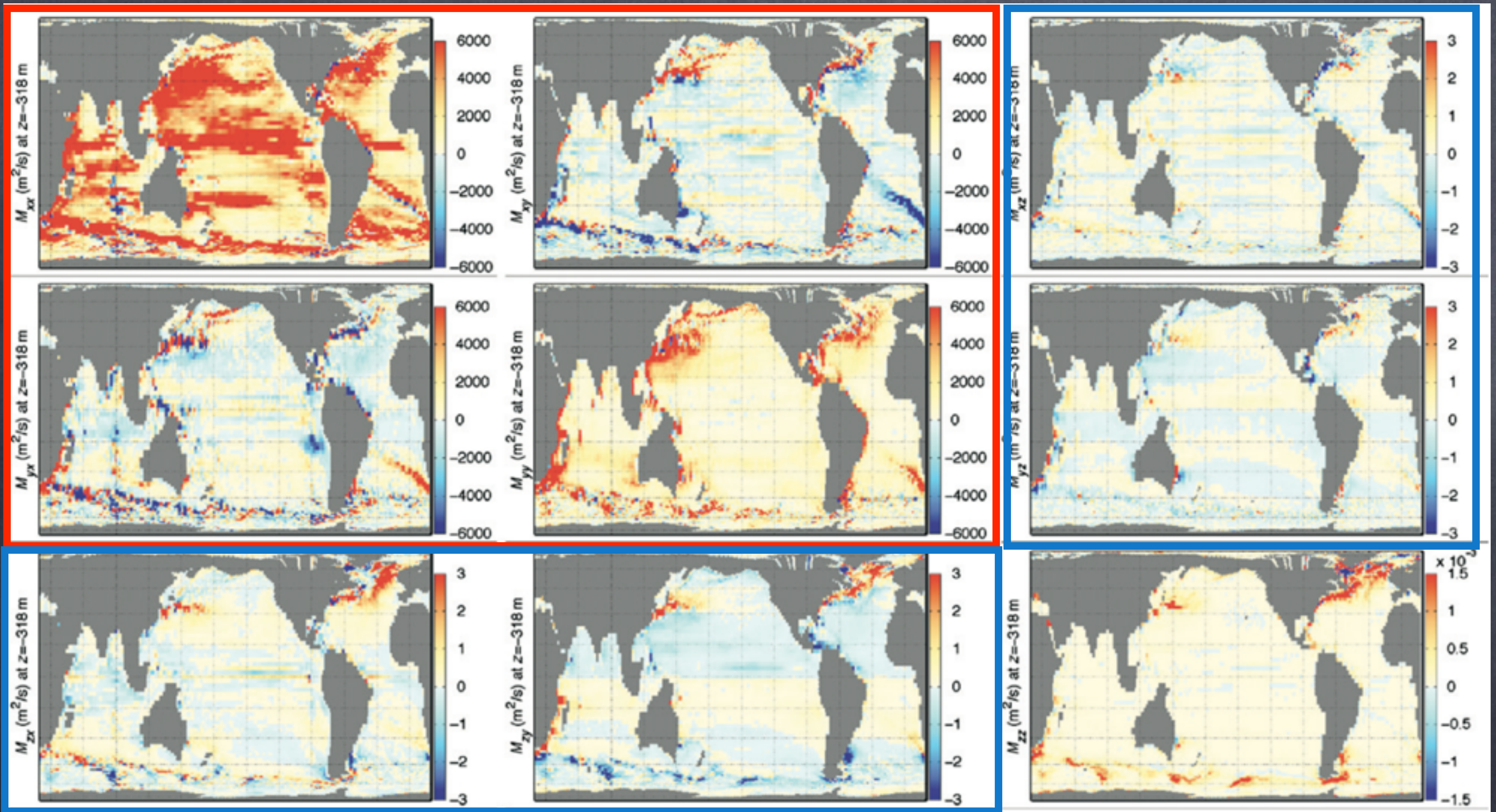


FIGURE 8.4 Components of the R tensor at 318 m depth, with the $K_{\alpha\beta}$ part in the upper left four panels.

Note: nearly symmetric in upper 2x2, NOT symmetric in outer row & column

BFK, R. Lumpkin, and F. O. Bryan. Lateral transport in the ocean interior. In G. Siedler, S. M. Griffies, J. Gould, and J. A. Church, editors, Ocean Circulation and Climate: A 21st century perspective, volume 103 of International Geophysics Series, chapter 8, pages 185-209. Academic Press (Elsevier Online), 2013.

What does (anti-)symmetry mean geometrically?

- Is the flux of a tracer down or up it's own gradient?
  - Diffusion $$(\nabla \tau_n) \cdot \mathcal{F}(\tau_n) = -\left(\partial_i \tau_n\right) S^{ij} \left(\partial_j \tau_n\right)$$
- Is the flux of *some* tracers down or up their own gradient, but others zero or small?
  - Anisotropic diffusion
- Is the flux of any tracer *never* down it's own gradient?
  - Advection $$(\nabla \tau_n) \cdot \mathcal{F}(\tau_n) = -\left(\partial_i \tau_n\right) A^{ij} \left(\partial_j \tau_n\right) = 0$$

It is critical to note that these hold not for one particular tracer, they hold true for any* tracer you consider with the same Lagrangian transport A and S.

Now—could this method be silly?

- If it fails objectivity—different in different coordinates. ✓
- It may be inconsistent with other intuition, e.g., mixing and stirring ✓
- If it is dependent on discretization ✓
- If it is dependent on details of averaging ✗
- It may be irrelevant in parameterizations ✓

Objective?: Change of Coordinates

- Any Cartesian
- Any orientation of axes

$$\partial_t\left(\rho\tau\right)+\partial_i\left[\rho\overline{u_i}\ \overline{\tau_n}+\rho\mathcal{F}(\tau_n)_i\right]=0,$$

where

$$-\mathcal{F}(\tau_n)_i=R_{ij}\partial_j\tau_n=\underbrace{A_{ij}\partial_j\tau_n}_{\text{advection}}+\underbrace{S_{ij}\partial_j\tau_n}_{\text{diffusion}},$$

$$A_{ij}=\frac{1}{2}\underbrace{\left(R_{ij}-R_{ji}\right)}_{\text{antisymmetric}},\qquad S_{ij}=\frac{1}{2}\underbrace{\left(R_{ij}-R_{ji}\right)}_{\text{symmetric}}.$$

- In Cartesian Coordinates:

Objective?: Change of Coordinates

- Any Orthogonal Curvilinear, such as cylindrical, spherical, or density, pressure, sigma, ALE as vertical coordinates

$$\partial_t(\rho\tau) + \partial_i\left[\overline{\rho u^i}\,\overline{\tau_n} + \rho\mathcal{F}(\tau_n)^i\right] = 0,$$

where

$$-\mathcal{F}(\tau_n)^i = R^{ij}\partial_j\tau_n = \underbrace{A^{ij}\partial_j\tau_n}_{\text{advection}} + \underbrace{S^{ij}\partial_j\tau_n}_{\text{diffusion}},$$

$$A^{ij} = \frac{1}{2}\underbrace{\left(R^{ij} - R^{ji}\right)}_{\text{antisymmetric}}, \qquad S^{ij} = \frac{1}{2}\underbrace{\left(R^{ij} - R^{ji}\right)}_{\text{symmetric}}.$$

- In Any Orthogonal Coordinates, and advection maps only to advection and diffusion only to diffusion.

Objective?: Change of Coordinates

- Any curvilinear coordinates, such as density, pressure, sigma, including metric curvature terms

That is, the covariant derivative including the Christoffel symbols preserves the symmetries.

$$\partial_t\left(\rho\tau\right) + \partial_{;i}\left[\overline{\rho u^i}\;\overline{\tau_n} + \rho\mathcal{F}(\tau_n)^i\right] = 0,$$

where

$$-\mathcal{F}(\tau_n)^i = R^{ij}\partial_j\tau_n = \underbrace{A^{ij}\partial_j\tau_n}_{\text{advection}} + \underbrace{S^{ij}\partial_j\tau_n}_{\text{diffusion}},$$

$$A^{ij} = \frac{1}{2}\underbrace{\left(R^{ij} - R^{ji}\right)}_{\text{antisymmetric}}, \qquad S^{ij} = \frac{1}{2}\underbrace{\left(R^{ij} - R^{ji}\right)}_{\text{symmetric}}.$$

- In Any Continuous & Differentiable Coordinates, and advection maps only to advection and diffusion only to diffusion.

Mixed, not Stirred

(on average, in averaged variables)

- Are symmetric and antisymmetric tensors distinct as mixing and stirring (Eckart)?
- Yes.

$$\frac{d}{dt}\int\left(\frac{\rho\tau^2}{2}\right)dV = \int\left(\underbrace{\rho\overline{u^j}(\partial_j\tau)(\tau)}_{\text{Eulerian Stirring}} + \underbrace{\cancelto{0}{\rho A^{ij}(\partial_i\tau)(\partial_j\tau)}}_{\text{Adv. Neutral}} + \underbrace{\rho S^{ij}(\partial_i\tau)(\partial_j\tau)}_{\text{Diff. Mixing}}\right)dV$$

(When integrated over whole domain, with no boundary sources)

Categorizing Parameterizations

- Gent-McWilliams 1990 is pure advection=anti-symmetric
- Redi 1982 is pure diffusion=symmetric
- Smith & Gent (2004) & Reckinger et al. are anisotropic diffusion & advection
- BFK et al. (2011) is pure advection
- Bachman & BFK (2013) extend (2011) to a combination of advection & diffusion
- Symmetric Instability of Bachman et al. is pure diffusion plus viscosity
- Fox-Kemper & Menemenlis (2008) QGLeith combines advection and isotropic diffusion

Depends on Averaging, Not Discretization

- We have seen already that it matters whether you are thickness-weighted, etc.

- We can objectively select a region for averaging, using a phase function from multi-phase or immersed boundary condition methods (Drew, 1983).

$$X_k(\mathbf{x}, t) = \begin{cases} 1 & \text{if } \mathbf{x} \text{ is in phase } k \text{ at time } t \\ 0 & \text{otherwise.} \end{cases}$$

It can be shown that

$$\frac{\partial X_k}{\partial t} + \mathbf{v}_i \cdot \nabla X_k = 0$$

in the sense of generalized functions.

Conclusions

- A diagnostic definition of Lagrangian advection and diffusion, based on simultaneous examination of multiple tracers is:
  - largely tracer-independent
  - objective (coordinate system invariant)
  - guiding parameterization development and evaluation
  - consistent with notions of mixing and stirring
  - able to be preserved under discretization
  - dependent on averaging, but in a mathematically precise way that can be made objective

In differential geometry terms,
we choose the most convenient gauge,
where the flux-gradient relation lives:

$$\partial_t \left(\rho \tau\right) + \partial_i \left[\rho \overline{u}\, \overline{\tau_n} + \rho \mathcal{F}(\tau_n)_i\right] = 0,$$

where

$$-\mathcal{F}(\tau_n)_i = R_{ij}\partial_j \tau_n = \underbrace{A_{ij}\partial_j \tau_n}_{\text{advection}} + \underbrace{S_{ij}\partial_j \tau_n}_{\text{diffusion}},$$

$$A_{ij} = \frac{1}{2}\underbrace{\left(R_{ij} - R_{ji}\right)}_{\text{antisymmetric}}, \qquad S_{ij} = \frac{1}{2}\underbrace{\left(R_{ij} - R_{ji}\right)}_{\text{symmetric}}.$$

- In Cartesian Coordinates (for the moment)
- Underdetermined, unless you use MULTIPLE TRACERS

S. Bachman and BFK. Eddy parameterization challenge suite. I: Eady spindown. Ocean Modelling, 64:12-28, April 2013.

S. Bachman, BFK, and F. O. Bryan. A tracer-based inversion method for diagnosing eddy-induced diffusivity and advection. Ocean Modelling, 86:1-14, February 2015.

Starting Point...