An Eddifying Parsons' Model

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I. Abstract

The time-mean effects of eddies are studied in a model based on the Parsons-Veronis-Huang-Fierl models of the wind-driven gyre. It is shown that much of the analysis used for the steady solutions carries over if cast in terms of the **thickness-weighted velocity**, *because the transport by this velocity is nondivergent in the absence of diabatic forcing. The model serves as a simple example of how the residual mean theory may simplify analysis in practice.*

A result of the analysis is a boundary layer width in the case of a rapid upper layer flow and weak lower layer flow. This boundary layer width is comparable to an eddy mixing length when the typical eddy velocity is taken to be the long Rossby wave phase speed.

II. Setup of the Parsons' Model

We consider a layered, rigid-lid, shallow-water, Boussinesq flow on a beta-plane:

$$\frac{\partial \boldsymbol{u}_i}{\partial t} + (\boldsymbol{f} + \boldsymbol{\zeta}_i) \times \boldsymbol{u}_i = -\boldsymbol{\nabla}\mathcal{B}_i + \boldsymbol{\mathcal{F}}_i, \tag{1}$$

$$\frac{\partial h_i}{\partial t} + \boldsymbol{\nabla} \cdot h_i \boldsymbol{u}_i = w_{i+} - w_{i-}. \tag{2}$$

The Bernoulli function $\mathcal{B}_i$ and Montgomery potentials, $\mathcal{M}_i$, for a 2-layer, rigid-lid model are:

$$\mathcal{B}_i \equiv \mathcal{M}_i + \frac{1}{2}\boldsymbol{u}_i \cdot \boldsymbol{u}_i, \quad \mathcal{M}_1 = g' h_1 + \mathcal{M}_2$$

Where g' is the reduced gravity ($g' \equiv g(\rho_2 - \rho_1)/\rho_1$). The total depth is fixed $h_1 + h_2 = H$. The lower layer can move if $\mathcal{M}_2 \neq 0$, but $\mathcal{M}_2$ is not simply related to layer depths (Cushman-Roisin, 1994, pp. 174-179).

Parsons (1969) showed that a steady-state, wind-driven solution can be found by neglecting inertia, motion in the lower layer ($\boldsymbol{u}_2 = 0$, $\mathcal{M}_2 = 0$), and mass exchange between layers ($w = 0$), as well as replacing the upper layer friction with wind forcing and interfacial drag:

$$\boldsymbol{f} \times h_1 \boldsymbol{u}_1 = -g' \boldsymbol{\nabla} \frac{h_1^2}{2} + \frac{\boldsymbol{\tau}}{\rho_0} - C_d \boldsymbol{u}_1,, \tag{3}$$

$$\boldsymbol{\nabla} \cdot h_1 \boldsymbol{u}_1 = 0. \tag{4}$$

If a time-mean adiabatic solution is sought instead a steady-state solution, different equations result from (1-2), including eddy correlations:

$$(\boldsymbol{f} + \overline{\boldsymbol{\zeta}}_i) \times \overline{\boldsymbol{u}}_i = -\boldsymbol{\nabla}\overline{\mathcal{B}}_i - \boldsymbol{\nabla}\overline{\mathrm{K}}_i + \overline{\boldsymbol{\mathcal{F}}}_i - \overline{\boldsymbol{\zeta}_i' \times \boldsymbol{u}_i'}, \qquad \boldsymbol{\nabla} \cdot \overline{h}_i \overline{\boldsymbol{u}}_i = -\boldsymbol{\nabla} \cdot \overline{h_i' \boldsymbol{u}_i'},$$

$$\overline{\mathcal{B}}_i \equiv \overline{\mathcal{M}}_i + \tfrac{1}{2}\overline{\boldsymbol{u}}_i \cdot \overline{\boldsymbol{u}}_i, \qquad \overline{\mathrm{K}}_i \equiv \tfrac{1}{2}\overline{\boldsymbol{u}_i' \cdot \boldsymbol{u}_i'}$$

Eddy correlations are conveniently handled as velocities. The Eulerian mean velocity $\overline{\boldsymbol{u}}_i$, the eddy-induced or bolus velocity $\overline{\boldsymbol{u}}_i^*$, and the thickness-weighted mean velocity $\overline{\boldsymbol{u}}_i^\dagger$ (McDougall and McIntosh, 2001) are defined and related by

$$\overline{\boldsymbol{u}}_i^\dagger \equiv \frac{\overline{h_i \boldsymbol{u}_i}}{\overline{h}_i}, \qquad \overline{\boldsymbol{u}}_i^* \equiv \frac{\overline{h_i' \boldsymbol{u}_i'}}{\overline{h}_i}. \qquad \overline{h_i \boldsymbol{u}_i} = \overline{h}_i \overline{\boldsymbol{u}}_i + \overline{h_i' \boldsymbol{u}_i'}, \qquad \overline{\boldsymbol{u}}_i^\dagger = \overline{\boldsymbol{u}}_i + \overline{\boldsymbol{u}}_i^*.$$

Rewriting with $\overline{\boldsymbol{u}}_1^\dagger$, and following the Parsons' model assumptions yields:

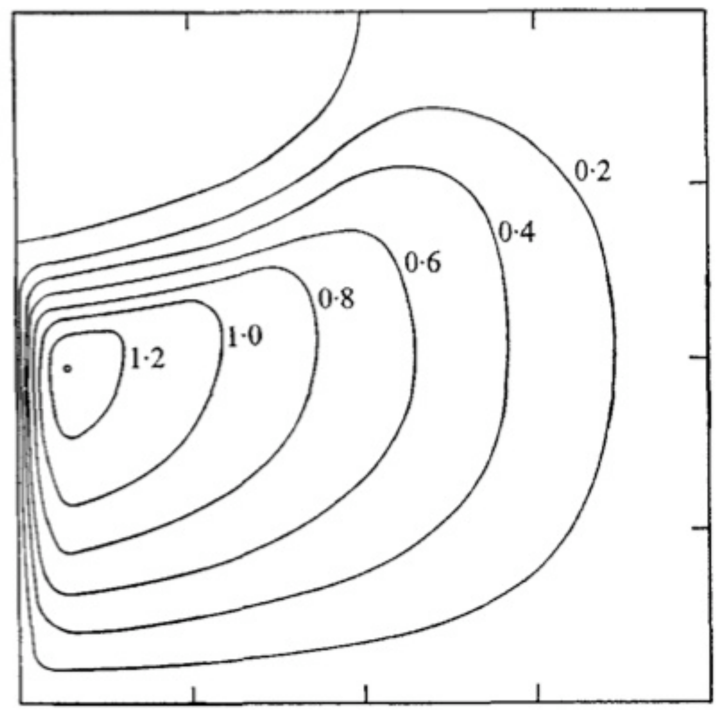
$$\boldsymbol{f} \times \overline{h}_1 \overline{\boldsymbol{u}}_1^\dagger = -g' \boldsymbol{\nabla} \frac{\overline{h}_1^2}{2} + \frac{\overline{\boldsymbol{\tau}}}{\rho_0} - C_d \overline{\boldsymbol{u}}_1 - \overline{h}_1^2 \hat{\boldsymbol{z}} \times \overline{P_1' \boldsymbol{u}_1'}, \tag{5}$$

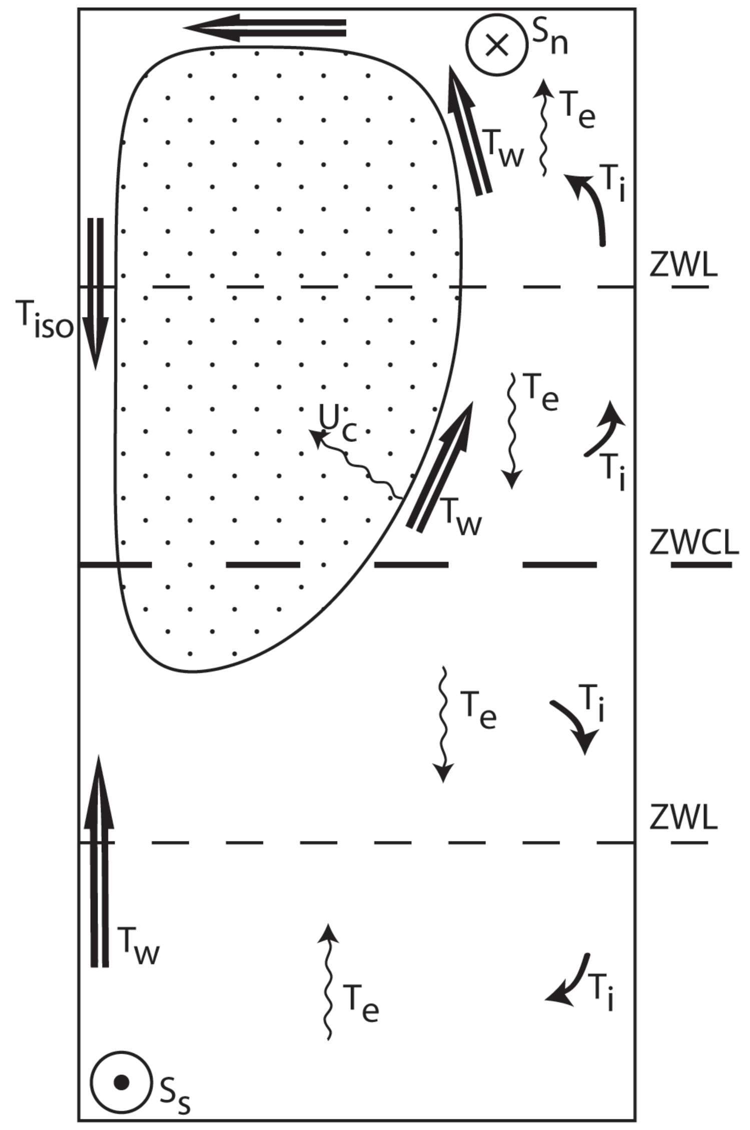
$$\boldsymbol{\nabla} \cdot \overline{h}_1 \overline{\boldsymbol{u}}_1^\dagger = 0. \tag{6}$$

Note the equivalence between (3-4) and (5-6)! The only difference is an eddy potential vorticity flux! With a relatively general for for a parameterization of this flux, we can proceed to solution.

$$\boldsymbol{f} \times \overline{h}_1 \overline{\boldsymbol{u}}_1^\dagger = -g' \boldsymbol{\nabla} \frac{\overline{h}_1^2}{2} + \frac{\overline{\boldsymbol{\tau}}}{\rho_0} - C_d \overline{\boldsymbol{u}}_1 - \kappa f \boldsymbol{f} \times \mathbf{A} \cdot \nabla \frac{\overline{h}_1}{f}. \tag{7}$$

IV. Solutions and Outcropping





Left: Streamfunction of the upper layer flow for a single-gyre wind demonstrating outcropping/boundary current separation (Parsons, 1969).

Right: A more complex model may be constructed using double-gyre winds and localized diabatic forcing (Veronis, 1976; Huang and Fierl, 1987; Nurser and Williams, 1990).

V. The Solution

We proceed with (5-6) and introduce a thickness-weighted mean transport streamfunction, $\overline{\Psi}_1^\dagger$. For the remainder of this section, we nondimensionalize following Parsons (1969) to yield:

$$-\frac{\partial \overline{\Psi}_1^\dagger}{\partial y} = \overline{h}_1 \overline{u}_1^\dagger, \qquad \frac{\partial \overline{\Psi}_1^\dagger}{\partial x} = \overline{h}_1 \overline{v}_1^\dagger,$$

$$f \boldsymbol{\nabla} \overline{\Psi}_1^\dagger = \boldsymbol{\nabla} \frac{\overline{h}_1^2}{2} - \lambda \boldsymbol{\tau} + \epsilon_i \overline{\boldsymbol{u}}_1 + \epsilon_e f \boldsymbol{f} \times \mathbf{A} \cdot \nabla \frac{\overline{h}_i}{f}. \tag{8}$$

Where $\lambda = LW/g'\rho_0 H_1^2$, $\epsilon_e = \kappa\beta L/g' H_1$, and $\epsilon_i = C_d/\beta L H_1$ are small numbers, roughly $5 \cdot 10^{-3}$, 10^{-3}, and $5 \cdot 10^{-6}$ based on typical ocean values (from Holland and Rhines, 1980). Thus, the interfacial drag is *negligible* in comparison to the eddy effects. Nonetheless, we may seek a solution to the full equations, and it is

$$\overline{h}_1 = \overline{h}_{1i} \frac{1 - Be^{\frac{-x}{\delta_b}}}{1 + Be^{\frac{-x}{\delta_b}}} \text{ for attached boundary layers, and} \tag{9}$$

$$\overline{h}_1 = \overline{h}_{1i} \frac{1 - e^{\frac{-(x - x_o)}{\delta_b}}}{1 + e^{\frac{-(x - x_o)}{\delta_b}}} \text{ for separated boundary layers, where} \tag{10}$$

$$\overline{h}_{1i} = \sqrt{\overline{h}_{1e}^2 - 2\lambda f^2 (1 - x) \frac{\partial(\tau/f)}{\partial y}}, \qquad B = \frac{\overline{h}_{1i} - \overline{h}_{1w}}{\overline{h}_{1i} + \overline{h}_{1w}}, \qquad \delta_b = \frac{\epsilon_i + \epsilon_e f^2 \mathbf{A}_{nn}}{\overline{h}_{1i} \frac{\partial f}{\partial s}}.$$

Where $\overline{h}_{1w}$ is the depth of the upper layer at the western boundary, and by semi-geostrophy is $\overline{h}_{1w}^2 = \overline{h}_{1i}^2 - 2f\overline{\Psi}_{1i}^\dagger = \overline{h}_{1i}^2 - 2f\lambda\frac{\partial \tau}{\partial y}$.

There are two key differences between the Parsons (1969) solution and the one here: 1) the use of thickness-weighted mean, and 2) the boundary layer width. For a comparison of relevant scales, the dimensional form of δ_b is

$$\delta_b = \frac{f^2 \kappa_1 \sin^2\theta}{g' \overline{h}_{1i} \frac{\partial f}{\partial s}}. \tag{11}$$

In terms of the deformation radius ($L_d = \sqrt{g'\overline{h}_{1i}}/f$), long Rossby wave speed ($c_R = \beta L_d^2$), and mixing length approximation ($\kappa_1 = L_e U_e$), this boundary layer width is comparable to the mixing length for eddies with $U_e = c_R$, multiplied by an $O(1)$ geometric factor.

$$\delta_b = \frac{\kappa_1}{c_R} \frac{\mathbf{A}_{nn}}{\hat{\mathbf{s}} \cdot \hat{\mathbf{y}}}. \tag{12}$$

Chelton et al. (2007) show that relevant eddy velocity scales are near to the Rossby wave speed.

VII. Conclusion

Incorporating the most basic effects of eddy fluxes into the Parsons model proves surprisingly easy. Only the boundary layer width and interpretation of the velocity field as a thickness-weighted mean is required. Interestingly, however, even this simple solution allows one to understand some of the important eddy-induced effects in the gyre circulation, e.g., downward transport of momentum by eddy form drag, scaling of the upper versus lower layer flow, etc.

The problem solved here is not accurate over a large regime–the neglect of inertial effects is a profound weakness not seen over most of the dominant regimes of numerical simulations–but nonetheless it is eddifying.

Acknowledgments

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