Tensor Time Derivatives for Fluid Dynamics in General Deforming Coordinate Systems

Joel Feske,[a] Baylor Fox-Kemper[a] , Jonathan M. Lilly[b]

[a] *Department of Earth, Environmental and Planetary Sciences, Brown University, Providence, Rhode Island*
[b] *Planetary Science Institute, Tucson, Arizona*

ABSTRACT: We begin the development of a tensor-based formalism for geophysical fluid dynamics using differential geometry to express the oceanic primitive equations in time-varying curvilinear coordinates. In this paper, the transformation from inertial to accelerating and rotating reference frames is made rigorous, systematically deriving apparent forces and sorting them into two classes: the familiar "fictitious" forces arising from rigid transformations, and the unfamiliar "fantastical" forces arising from non-rigid, deforming transformations. Though the coordinate systems chosen in ocean modeling are often deforming, these latter apparent forces are often overlooked. This work and companion papers provide a unifying mathematical foundation that improves communication among ocean modelers, permits higher-order, multi-dimensional numerical schemes, and forms a basis for the development of next-generation ocean models and parameterizations.

1. Introduction

Geophysical fluid dynamics (GFD) is a non-relativistic branch of fluid dynamics in the Newtonian limit of gravity, centered on the effects of stratification, rotation, and topography. These features break the isotropy of fluid motion, privileging flow, stirring, and mixing in certain directions and within particular curved surfaces, for example isosurfaces of neutral buoyancy. Numerical models exploit the geometry of broken symmetries via dynamically motivated choices of coordinates, such as a vertical direction aligned with effective gravity and thus hydrostatic balance. Such coordinate choices simplify equations and increase computational efficiency.

Coordinate systems, however, do not exist. This is what makes them so useful: they can be chosen for convenience in the problem at hand. Nature prescribes the quantities—density, momentum, strain—that determine the state of a system, along with the laws by which the system evolves, and these physical quantities themselves are coordinate-free. This freedom from coordinate dependence is what we mean when we say that such quantities are tensor fields: all choices of their representations—all sets of components—correspond to the same object in nature. Technical definitions of tensors as transforming in certain ways under a change of coordinates or as multilinear functions from vectors to scalars are all constructed to ensure that we maintain a sharp separation between the tensor itself and our representation of that tensor's components.

Working with tensors in generalized coordinates, we can on the one hand choose any representation of nature that suits our problem, and on the other hand ensure that we are dealing with real, physical quantities that are not simply artifacts of our coordinate choice. Tensor equations that apply under all coordinates are particularly useful in geophysical fluid dynamics where we use space and time-varying coordinates to capture planetary rotation, and often also choose time-varying coordinates to simplify the dynamics of stratification.

Tensor analysis has the power to express the equations of fluid motion without approximation in as general a choice of coordinate system as possible. This includes familiar rigidly rotating and translating coordinate systems as well as potentially deforming systems, such as isopycnal or terrain-following coordinates. Our first goal is to define a component-wise time derivative $\delta/\delta t$ which, when applied to the components of a tensor $\boldsymbol{T}$ in some basis, provides the components of the time derivative of the tensor $d\boldsymbol{T}/dt$ in the same basis. Such an operation must permit both spatial and temporal dependence of the coordinate bases linked to those components. The *intrinsic time derivative*, discussed in Aris (1962) and Grinfeld (2013) accomplishes this goal for spatially varying basis vectors, however, its definition presumes that for a given location the basis is fixed in time; this is virtually never the case in geophysical fluid dynamics. The *generalized intrinsic time derivative* introduced here extends this idea to permit time-dependent basis vectors.

Changes observed in the tensor field $\boldsymbol{T}$ depend not only on physical processes, but also on the observer's path through it, and the generalized intrinsic time derivative captures both types of change. If the observer's location tracks a fluid parcel, as in the Lagrangian fluid equations, the generalized intrinsic derivative reduces to a generalized intrinsic *material* derivative. With this material derivative finally in hand, we proceed to give expressions for the components of inertial velocity and acceleration in all coordinate systems, including rigidly rotating and translating but also deforming systems. These expressions include the familiar advection terms and "fictitious forces"—the Coriolis and centrifugal forces—but also include other,

Corresponding author: Joel Feske, joel_feske@brown.edu

unfamiliar apparent forces that arise only in deforming systems. With the choice of Lagrangian coordinates, we connect these new terms to the convected time derivative of Oldroyd (1950) and to the Lie derivative that is so useful in geometric mechanics (Holm 2025).

In GFD, different temporally and spatially dependent coordinate systems, often quite complex, are chosen for convenience in different situations. Most fundamentally, non-inertial frames of reference are typically used to separate the large velocities of planetary motion from the relative velocities more easily measured by earthbound observers. Similarly, deforming coordinates are often used to capture the anisotropic flows along the most natural surfaces.

In physical oceanography, for example, coordinate choices generally depend on the region of interest (Fox-Kemper et al. 2019): near the surface and in regions of weak stratification, height coordinates are preferred; near the sea floor, topography drives dynamics and terrain-following, or σ, coordinates are preferred; and away from the boundaries in regions of strong stratification, isopycnal coordinates (McDougall 1987; Young 2012) are typically chosen to capture the undulation of density surfaces. More generally, Lagrangian coordinates (e.g., Salmon 2013) flow with a fluid while Eulerian systems stay fixed in coordinate space as the fluid moves through them. The Arbitrary Lagrangian-Eulerian method (e.g., Petersen et al. 2015; Gibson et al. 2017; Megann et al. 2022) and other complex coordinates as used in the HYCOM (Bleck 2002; Halliwell 2004) and MOM6 (Griffies et al. 2020) numerical models combine these approaches, allowing the grid to flow with the fluid for a time, resetting before it becomes too deformed.

The choice of coordinate system has real relevance for the accuracy of numerical models. Finite machine precision can introduce error into simulations, and an improper choice of coordinates will exacerbate that error. For example, is has long been known that modeling tracer transport in height coordinates leads to the Veronis effect: if lateral transport is calculated perpendicular to the geopotential gradient at constant height, rather than more realistically along surfaces of constant density, the misalignment of these two sorts of surfaces causes spurious mixing across density classes (Veronis 1975). The difference of roughly eight orders of magnitude between isopycnal and diapycnal mixing in the real ocean means that small errors in the representation of isopycnal surfaces introduce significant spurious diapycnal mixing—if even one one-millionth of the isopycnal mixing is erroneously rotated into the diapycnal direction, it will overwhelm the true diapycnal mixing. More suitable choices of coordinates can help to avoid this (Griffies et al. 2000, 2015; Ilıcak et al. 2012).

Coordinate-free formulations based on tensors find application in numerical methods for modeling fluid flows for other reasons as well. An important class of totally anti-symmetric tensors, dubbed *differential forms*, are especially useful for integration and discretization, particularly on unstructured meshes (Crane et al. 2013; Crane 2018), and Cotter and Thuburn (2014) have applied these methods to the rotating shallow water equations. Åhlander et al. (2001), meanwhile, note that "the design of applications for scientific computing should be based firstly upon continuous abstractions and secondly on approximations," and show that separating software engineering concerns by the level of mathematical abstraction allows discretization to occur more naturally at each level. The literature on the Sophus software library details many of these discrete, tensor-based methods (Grant et al. 2000; Engø et al. 2001; Friis et al. 2001; Haveraaen et al. 2005; Åhlander and Otto 2006; Haveraaen and Friis 2009; Heinzl and Schwaha 2011; Gilbert and Vanneste 2023).

There is plenty of precedent for the application of tensor analysis to fluid mechanics in other domains, most obviously in relativistic fluids where spacetime itself—not just the coordinate system—is so curved as to require careful treatment. The black hole accretion study by Hawley et al. (1984) is exemplary. It is therefore natural to introduce a time derivative to non-relativistic dynamics by taking the limit $v \ll c$, where $c \approx 3 \times 10^8$ m s^{-1} is the speed of light. Maddison and Marshall (2013) take this approach to deriving the Eliassen-Palm flux tensor in time-varying coordinates, and the approach also appears in Griffies (2004). However, this introduces a speed-of-light velocity scale c that is cumbersome alongside atmospheric winds and ocean currents with mean velocities roughly six to ten orders of magnitude smaller. We avoid this by taking a non-relativistic approach from the start, which has the added benefit of grounding the problem in familiar, flat 3D space.

Putting aside relativistic physics, introductory fluids texts (e.g., Kundu et al. 2016) often frame equations in Cartesian tensor notation with a fixed basis $\{\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}\}$. Crucially for our purposes, partial derivatives of tensor components with respect to spatial coordinates do not immediately generalize to deforming, curvilinear coordinates, but require the added algebraic structure of the *covariant derivative*, discussed subsequently. To fully generalize component-wise tensor computations to non-orthogonal coordinates, components must be tracked with respect to both a coordinate basis $\boldsymbol{e}_i$ and its dual or reciprocal basis $\boldsymbol{e}^i$; these happen to coincide in Cartesian coordinates, resulting in a considerable simplification.

This work builds on the rich history of advances in tensor analysis in fluid mechanical. Tensor analysis for geophysical fluids is reviewed in Griffies (2004, Chapter 18) and has been applied to the Gent-McWilliams skew flux (Griffies 1998; Dukowicz and Smith 1997) and to thickness-weighted averaging (Young 2012). Oldroyd (1950) employs tensor analysis to define time derivatives for materially advected coordinates, although in quite a different sense than we do, and Hinch and Harlen (2021) provide relevant context for Oldroyd's work. Grinfeld (2009) employs tensor analysis in a fully curvilinear treatment of thin fluid films, which Grinfeld (2013) expands into the full calculus of moving surfaces.

The goal of this work is three-fold. Firstly, we aim to create a primer on the tensor analysis of Euclidean space in curvilinear coordinates, to highlight the remarkable power and compactness of index notation, and to provide tensor component-based equations that can be directly translated into model code. This approach offers high levels of both abstraction and practicality, since any expression written in index notation is immediately computable. Secondly, we construct the generalized intrinsic time derivative discussed above. Finally, we use this derivative to investigate the apparent forces that arise in the general, non-inertial, deforming coordinate systems so common in GFD. These apparent forces separate cleanly into two categories based on tensor symmetries: the familiar fictitious forces arising from the rigid (translational and/or rotational) part of the coordinate transformation, and what we dub the *fantastical forces* arising from the non-rigid (straining and/or diverging) part. The fantastical forces are more subtle than the fictitious forces because they differ for covariant and contravariant representations of a tensor field.

The structure of this paper is as follows: sections 2 and 3 lay out the definitions, notation, and conventions that we employ for the tensor calculus of Euclidean space; section 4 derives the form of the generalized time derivative; section 5 applies it to the momentum equation and shows how to recover the familiar rotating equations of motion; and section 6 discusses generalizations of the fictitious forces that arise from non-rigid motion, provides more a abstract form of the intrinsic derivative, and discusses its connection to related work.

2. Preliminaries: definitions, notation, and conventions

This section introduces the definitions, notation, and conventions of tensor analysis employed throughout. Prior familiarity with tensor analysis is assumed; readers already proficient in tensor analysis using index notation can skip to section 3 where the GFD-specific aspects begin. For extensive treatments of this topic, the authors favor Grinfeld (2013), Carroll (2004), Itskov (2015), Aris (1962), and Ogden (1984); see also Lilly et al. (2024). For brevity, many statements are given without proof, as they are readily found in the literature.

One of the goals of this work, and of this section specifically, is to be a brief introductory reference for tensor analysis and differential geometry as needed for GFD applications, in particular numerical modeling of the ocean circulation in moving coordinate systems. This leads us to make two fundamental choices: *extrinsic geometry*, meaning that the existence of flat, three-dimensional Euclidean, inertial space may be presumed, and *explicit notation*, in which we use index-based notation for tensor analysis so that mathematical expressions can be readily converted into computer code for numerical calculations. Our presentation aims not only to introduce useful abstractions but also to eschew needless ones. Although many complementary mathematical systems (such as exterior algebra) exist and are sometimes more, and sometimes less, direct that our approach. We occasionally refer back to equations in their symbolic form when index notation becomes onerous to interpret.

a. Space, coordinate systems, and basis vectors

In non-relativistic three-dimensional Euclidean space, the setting for GFD, time is a parameter shared by all spatial locations. A *scalar field* is a distribution of values at each point in space. A *vector field* can be visualized as a distribution of arrows, having magnitude and direction, at each point in space. Scalar and vector fields respectively are tensor fields of order zero and one. A *second-order tensor field* is a linear mapping from vectors to vectors, or a bilinear mapping from pairs of vectors to scalars, at each point in space. More generally, an nth order tensor field is a multilinear function from n vectors to scalars, although in GFD we rarely encounter tensors higher than second order. As all tensors encountered are tensor fields, spatial and temporal dependence of these entities is generally understood.

A *coordinate system* is a set of enumerated surfaces extending throughout space, not necessarily planar nor spatially fixed in time, that can be used to specify locations and therefore to quantify motion. These coordinate surfaces are indexed with a set of three numbers $\{x^1, x^2, x^3\}$ called the *coordinates*. When used as the argument to a function, the shorthand notation $f(x)$ will be used in place of $f(x^1, x^2, x^3)$, following Grinfeld (2013). The *position vector field* $\boldsymbol{r}(x)$ of the coordinate system is the vector field that at each coordinate location $\{x^1, x^2, x^3\}$ gives the displacement vector from the origin to that location. For notational compactness, the argument x will generally be omitted from $\boldsymbol{r}$ and other tensor fields. We shall consider x^i to refer collectively to the entire set $\{x^1, x^2, x^3\}$, and similarly for other indexed entities.

Critically, physical laws, and the tensors appearing in them, are not affected by the choice of coordinate systems. However, the representation of tensors as sets of components and the representation of physical laws as equations expressed in terms of these tensor components *are affected* by the choice of coordinate system. Much of the effort expended here is in realizing this distinction in equations for tensor components, because these component equations are what are actually used for calculation in numerical models. In our experience this distinction is often confounded in GFD discussions, because a rotating coordinate system encodes both the change of coordinates and the change to a rotating frame of reference into the apparent forces that result from the non-inertial frame.

Differentiation of the position vector field $\boldsymbol{r}$ with respect to each coordinate defines a set of *basis vectors*

$$\boldsymbol{e}_i \equiv \partial_i \boldsymbol{r} \; , \tag{1}$$

where $\partial_i \equiv \partial/\partial x^i$ is a shorthand for the partial derivative with respect to the ith coordinate. A basis vector is thus the tangent vector to the curve traced out by the change in position as we vary $\boldsymbol{r}$ one coordinate x^i at a time. The basis vectors need not have unit length, be orthogonal to one another, or be constant in space and time. Singularities in the coordinate systems can also be managed, for example at extrema in non-monotonic coordinates and at poles (see, e.g., Prusa 2018), so long as there are not too many.

In order to handle so many possibilities for the basis $\boldsymbol{e}_j$, a set of *reciprocal basis vectors* $\boldsymbol{e}^i$ are defined such that

$$\boldsymbol{e}^i \cdot \boldsymbol{e}_j \equiv \delta^i_j \; , \tag{2}$$

with δ^i_j being the Kronecker delta function

$$\delta^i_j \equiv \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases} \tag{3}$$

Unlike in Cartesian coordinates, the $\boldsymbol{e}^i$ and $\boldsymbol{e}^j$ bases need not be orthonormal, since curvilinear coordinate bases are neither orthogonal nor normalized in general. The reciprocal basis helps this manage by introducing a new type of joint orthonormality.

On account of their behavior under coordinate transformations, the $\boldsymbol{e}_i$ and $\boldsymbol{e}^i$ are also called the *covariant* and *contravariant* basis vectors, respectively. Note that covariant vectors are indicated with a lowered index while contravariant vectors are indicated with a raised index. If coordinate surfaces are varied to become more tightly packed, vectors of the former type shrink to co-vary, while those of the latter type stretch to contra-vary. For those familiar with tensors in Cartesian coordinates, but unfamiliar with tensors in general curvilinear coordinates, the distinction between these two types of vectors is a key new aspect to consider, and critical for understanding the implications of deforming coordinates for GFD.

In what follows we will make use of the partial derivatives of the basis and reciprocal basis vectors with respect to the coordinates. In flat space, these can be expressed as

$$\partial_i \boldsymbol{e}_j = \Gamma^k{}_{ij} \boldsymbol{e}_k \; , \qquad \partial_i \boldsymbol{e}^j = -\Gamma^j{}_{ik} \boldsymbol{e}^k \; , \tag{4}$$

where $\Gamma^k{}_{ij}$, the *Christoffel symbol of the second kind*, is defined as

$$\Gamma^k{}_{ij} \equiv \left(\partial_i \boldsymbol{e}_j\right) \cdot \boldsymbol{e}^k. \tag{5}$$

Thus $\Gamma^k{}_{ij}$ is the expansion coefficient for the derivative of the jth basis vector with respect to the ith coordinate that is associated with the kth basis vector. One may readily show the symmetry $\Gamma^k{}_{ij} = \Gamma^k{}_{ji}$ from (1) and the commutativity of partial derivatives. A readily computable expression for the Christoffel symbols will be presented shortly.

The extrinsic approach to the formulation of tensor analysis employed here offers important simplifications over the *intrinsic* approach common in general relativity. There, all definitions are intrinsic to the curved spacetime manifold, as knowledge of any extrinsic space is not observable. With an extrinsic space we can think of basis vectors $\boldsymbol{e}_i$ simply as spatially and temporally varying arrows in physical space rather than solely as differential operators in the tangent space of the spacetime manifold, as is the case in intrinsic formulations of general relativity. Furthermore, here the basis vectors $\boldsymbol{e}_i$ and reciprocal basis vectors $\boldsymbol{e}^i$ are elements of the same Euclidean vector space, whereas in general relativity the reciprocal basis $\boldsymbol{e}^i$ must be replaced with the dual basis of differential 1-forms in the cotangent space to the manifold. See section 2.4 of Carroll (2004) or section 1.4 of Ogden (1984) for further discussion.

The general, time-varying, curvilinear coordinate system x^i captures both the rotational acceleration and deformation required for GFD. In addition, we introduce a second, potentially curvilinear but rigid and inertial, coordinate system y^a. On account of Galilean invariance we are free to choose an absolute reference for motion such that this inertial coordinate system is regarded to be stationary. A sketch is presented in Fig. 1, which also includes notation introduced subsequently. In order to clearly distinguish between tensor components referenced to these two systems, we adopt the convention of indexing any object as referenced from the non-inertial, deforming setting with the letters $\{i, j, k, \ell, m\}$ and from the inertial system with $\{a, b, c, d, e\}$. The less common choices of letters in the latter case are intended to reflect the less common choice of an inertial coordinate system for GFD applications.

Let $\boldsymbol{s}$ be the position vector field associated with the y^a coordinate system, with $\boldsymbol{s} = \boldsymbol{r} + \boldsymbol{S}$ where $\boldsymbol{S}$ gives the location of origin O_x of the x^i system with respect to the origin O_y of the y^a system; see Figure 1. The y^a system has covariant basis vectors

$$\boldsymbol{e}_a \equiv \partial_a \boldsymbol{s} = \partial_a \boldsymbol{r} \; , \tag{6}$$

where $\partial_a \equiv \frac{\partial}{\partial y^a}$, and where the latter equality follows from the fact that $\boldsymbol{S}$ is spatially constant. Note that the basis vectors $\boldsymbol{e}_i$ and $\boldsymbol{e}_a$ are distinguished by their indices, as are the partial derivatives ∂_i and ∂_a, the Christoffel symbols $\Gamma^k{}_{ij}$ and $\Gamma^c{}_{ab}$, and so forth. While the curvilinear coordinates in the inertial y^a system may have complex spatial dependence, the time derivatives of these coordinates, their basis vectors, and any other quantities such as $\boldsymbol{s}$ and Christoffel symbols are all temporally constant. In the x^i system, these quantities may all vary in time.

b. Index notation

In Euclidean space a vector $\boldsymbol{w}$ may be equivalently represented in terms of the basis or the reciprocal basis as

$$\boldsymbol{w} = w^i \boldsymbol{e}_i = w_i \boldsymbol{e}^i \,, \tag{7}$$

where (2) is used to find the components as $w^i \equiv \boldsymbol{w} \cdot \boldsymbol{e}^i$ and $w_i \equiv \boldsymbol{w} \cdot \boldsymbol{e}_i$, called the *contravariant* and *covariant* components of $\boldsymbol{w}$, respectively. Note that herein we use the Einstein convention that any indices occurring in one upper and one lower position are understood to be summed over, so that $w^i \boldsymbol{e}_i$ means $\sum_{i=1}^{3} w^i \boldsymbol{e}_i$. For the special case of a Cartesian basis, the covariant and contravariant basis vectors are identical, and consequently so are the covariant and contravariant components. In curvilinear coordinates, however, the index position is essential in order to keep track of which set of components is being considered.

Similarly, a second-order tensor $\boldsymbol{A}$ can be variously represented variously in terms of its contravariant, covariant, or mixed components as

$$\begin{aligned}\boldsymbol{A} = A^{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j = A_{ij} \boldsymbol{e}^i \otimes \boldsymbol{e}^j \\ = A^i{}_j \boldsymbol{e}_i \otimes \boldsymbol{e}^j = A_i{}^j \boldsymbol{e}^i \otimes \boldsymbol{e}_j,\end{aligned} \tag{8}$$

where $\otimes$ denotes the tensor product of the basis vectors, the proper vector analogue of the familiar outer product of linear algebra. Each of these expressions involves a sum over nine elements, with two nested sums over three basis vectors in each sum. As with vector components, the tensor components can be found by projecting $\boldsymbol{A}$ onto the basis vectors, $A^{ij} \equiv \boldsymbol{e}^i \cdot \left(\boldsymbol{A}\boldsymbol{e}^j\right)$ and so forth, again using orthogonality (2) together with the defining property

$$(\boldsymbol{a} \otimes \boldsymbol{b})\, \boldsymbol{w} = \boldsymbol{a}\, (\boldsymbol{b} \cdot \boldsymbol{w}) \tag{9}$$

of the tensor product (see Itskov 2015, section 1.7).

The index state of a tensor component—its number of indices together with their status as contravariant or covariant—is known as its *valence* (e.g., Needham 2021, §33.1). Unlike the vector components w^i, the coordinates x^i are not the components of some vector field $\boldsymbol{x}$ as the notation appears to imply; they are just the numbers specifying the location of a particular point with respect to the chosen coordinate system. In particular, they are not the same as the components r^i of the position vector field $\boldsymbol{r}$.

We adopt the convention that the left-to-right order of the indices in expressions such as $A^{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$ must match between the components and the basis. With this convention, it becomes unnecessary to explicitly indicate the basis vectors that accompany any tensor's components, since their arrangement is evident from placement of the indices within the components. That is, the tensor component A^{ij} may be used to unambiguously refer to the entire tensor $A^{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$. Consequently, one need not write the basis vectors at all in evaluating tensor operations, just the relations among components. However, by the product rule the derivatives of tensors depend on both the derivatives of the components and the derivatives of the basis vectors. It follows that in order to dispense entirely with basis vectors some special machinery for expressing derivatives in terms of tensor components is needed, as introduced shortly.

c. Index juggling

The basis vectors and reciprocal basis vectors can be used to define the *metric coefficients*

$$g_{ij} \equiv \boldsymbol{e}_i \cdot \boldsymbol{e}_j, \qquad g^{ij} = \boldsymbol{e}^i \cdot \boldsymbol{e}^j \tag{10}$$

which are symmetric, are not equal to δ^i_j except in the special case of orthonormal basis sets. With these definitions, the contravariant and covariant components of the vector $\boldsymbol{w} = w^i \boldsymbol{e}_i = w_i \boldsymbol{e}^i$ can converted into one another. This conversion proceeds as

$$w_j g^{ji} = w_j \boldsymbol{e}^j \cdot \boldsymbol{e}^i = \boldsymbol{w} \cdot \boldsymbol{e}^i = w^j \boldsymbol{e}_j \cdot \boldsymbol{e}^i = w^j \delta^i_j = w^i \tag{11}$$

$$w^j g_{ji} = w^j \boldsymbol{e}_j \cdot \boldsymbol{e}_i = \boldsymbol{w} \cdot \boldsymbol{e}_i = w_j \boldsymbol{e}^j \cdot \boldsymbol{e}_i = w_j \delta^j_i = w_i \tag{12}$$

which together establish that

$$g^{ij} w_j = w^i, \qquad g_{ij} w^j = w_i. \tag{13}$$

Using the metric coefficients to convert between covariant and contravariant components is called *index raising and lowering*, *index juggling*, or *changing valence*. Note that index raising inverts index lowering, and thus $g^{ij} g_{jk} = \delta^i_k$.

d. Tensor operations

When performing operations on tensors, one has the choice of working in symbolic notation, using familiar operators such as $\times$, $\otimes$, and $\boldsymbol{\nabla}$, or in index notation in which one directly represents these operations through their impact on tensor components. Translations of common operations and identities between these two notational system are presented in Tables 1–3. See section 1 of Itskov (2015) or Lilly et al. (2024) for the symbolic notation definitions of the tensor product $\boldsymbol{a} \otimes \boldsymbol{b}$, left operation $\boldsymbol{a}\boldsymbol{A}$, the tensor transpose $\boldsymbol{A}^T$, and the tensor trace $\mathrm{tr}\,(\boldsymbol{A})$, all of which are perfectly analogous to their familiar analogues in linear algebra, but which may be defined in the abstract without reference to any tensor components. Note also that as $\boldsymbol{a} \cdot (\boldsymbol{A}\boldsymbol{b}) = (\boldsymbol{a}\boldsymbol{A}) \cdot \boldsymbol{b}$, one may write this simply as $\boldsymbol{a}\boldsymbol{A}\boldsymbol{b}$.

In symbolic notation, the action of the tensor $\boldsymbol{A}$ on the vector $\boldsymbol{w}$ is represented as

$$\boldsymbol{A}\boldsymbol{w} = A^i{}_j \left(\boldsymbol{e}_i \otimes \boldsymbol{e}^j\right) w^k \boldsymbol{e}_k = A^i{}_j w^j \boldsymbol{e}_i \tag{14}$$

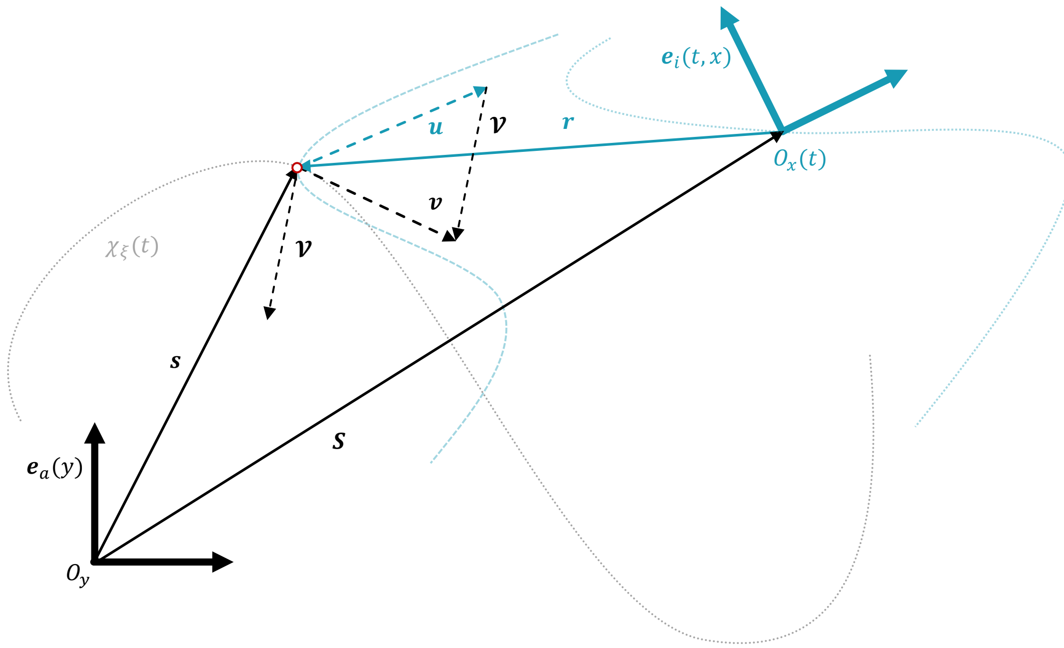

FIG. 1. A schematic showing quantities in an inertial frame (black), chosen without loss of generality to be stationary, and a general moving, deforming coordinate system as often used in GFD (teal). Heavy solid lines indicate coordinate axes, thin solid lines are position vectors, thin dashed lines are velocity vectors, and dashed or dotted curves are trajectories. The inertial coordinate system $y^a = \{y^1, y^2, y^3\}$ has basis vectors $\boldsymbol{e}_a$ and fixed origin O_y, while the moving coordinate system $x^i = \{x^1, x^2, x^3\}$ has basis vectors $\boldsymbol{e}_i$ and moving origin $O_x(t)$, the path of which is indicated by the dotted teal curve. The instantaneous displacement at some time t_0 from the fixed origin O_y to the moving origin $O_x(t)$ is given by the vector $\boldsymbol{S}$. Addition information on this plot is not used until later sections. Then, we will consider a parameterized path $\chi_\xi(t)$, representing the time-varying position of a fluid parcel labeled ξ, the position of which at t_0 is indicated by the small white circle. The position vectors of this parcel relative to O_y and $O_x(t)$ at time t_0 are $\boldsymbol{s}$ and $\boldsymbol{r}$ respectively, which satisfy $\boldsymbol{s} = \boldsymbol{r} + \boldsymbol{S}$. At time t_0, the fluid parcel has velocity $\boldsymbol{v}$ as observed from the inertial y coordinate system and velocity $\boldsymbol{u}$ as observed from the moving, deforming x coordinate system. Through a combination of the motion of $O_x(t)$ and the internal motion and deformation of the x coordinate system, the x coordinate location occupied by the fluid parcel at time t_0 also traces out a curve in space, indicated by the dashed teal curve. At time t_0, the velocity of this x coordinate location as viewed from the inertial frame is the vector $\mathcal{V}$, which is necessarily tangent to the curve. Note that $\boldsymbol{e}_i$ is a function of time *and space*, so each location in space has a potentially different basis vector set. The velocities are related by $\boldsymbol{v} = \boldsymbol{u} + \mathcal{V}$, as shown. While we have illustrated this with an individual parcel, $\boldsymbol{v}$, $\boldsymbol{u}$, and $\mathcal{V}$ are vector fields which add in this way at each point in the fluid.

again using (9) and (2). But because the presence of the $\boldsymbol{e}_i$ can be inferred, we can write down $A^i{}_j w^j$ as an implicit representation of $\boldsymbol{A}\boldsymbol{w}$ without needing to carry out the intervening manipulations on the basis vectors. The action of matching an upper with a lower index in two tensor entities, as in $A^i{}_j w^j$, is called a *contraction*, and with the Einstein convention a contraction can be spotted easily by a repeated index in an upper and lower position in a single term or product. Each contraction involves the implicit dot product of a basis vector $\boldsymbol{e}_i$ and a reciprocal basis vector $\boldsymbol{e}^j$ using orthogonality (2), reducing the degree of the tensor by two. Index notation can readily accommodate multiple contractions at the same time across many indices, which are easily identified by matching indices.

A particularly important tensor is the *identity tensor*, which in flat Euclidean space is given by

$$\boldsymbol{I} = g^{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j = g_{ij}\boldsymbol{e}^i \otimes \boldsymbol{e}^j = \delta^i_j \boldsymbol{e}_i \otimes \boldsymbol{e}^j = \delta^j_i \boldsymbol{e}^i \otimes \boldsymbol{e}_j. \quad (15)$$

Its non-mixed components are the metric coefficients, while its mixed components are Kronecker delta functions. This tensor has the property that $\boldsymbol{I}\boldsymbol{w} = \boldsymbol{w}$ for all $\boldsymbol{w}$. The identity tensor is implicitly involved whenever the delta function is used in index notation, that is, $a_i \delta^i_j b^j = a_i b^i$ corresponds to the tensor equation $\boldsymbol{a}\boldsymbol{I}\boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{b}$ with $\boldsymbol{I}$ chosen in a mixed representation. It is also implicitly involved in index juggling, since $g^{ij} w_j = w^i$ corresponds to the tensor equation $\boldsymbol{I}\boldsymbol{w} = \boldsymbol{w}$ with $\boldsymbol{I}$ chosen in a non-mixed representation. This is intuitive, because changing the representation of tensor components is an operation that does not change the tensor itself, that is, an identity operation.

TABLE 1. Common vector operations, written both in symbolic notation and in index notation, with the basis vectors associated with the index notation expressions shown in the final column. Here $\boldsymbol{\nabla} \equiv \boldsymbol{e}^i \partial_i$ denotes the familiar nabla or del operator as expressed in curvilinear coordinates, $\otimes$ denotes the tensor product, ∇_i is the covariant derivative acting on tensor components, and ε^{ijk} is the Levi-Civita or alternating tensor, all of which are defined subsequently.

Name	Symbolic	Index	Index Basis
Inner product	$\boldsymbol{a} \cdot \boldsymbol{b}$	$a_i b^i$	—
Cross product	$\boldsymbol{a} \times \boldsymbol{b}$	$\varepsilon^{ijk} a_j b_k \boldsymbol{e}_i$	$\boldsymbol{e}_i$
Tensor product	$\boldsymbol{a} \otimes \boldsymbol{b}$	$a_i b^j$	$\boldsymbol{e}^i \otimes \boldsymbol{e}_j$
Gradient	$\boldsymbol{\nabla} \gamma$	$(\nabla_i \gamma)$	$\boldsymbol{e}_i$
Divergence	$\boldsymbol{\nabla} \cdot \boldsymbol{w}$	$\nabla_i w^i$	—
Curl	$\boldsymbol{\nabla} \times \boldsymbol{w}$	$\varepsilon^{ijk} \nabla_j w_k$	$\boldsymbol{e}_i$
Gradient tensor	$\boldsymbol{\nabla} \otimes \boldsymbol{w}$	$\nabla_i w^j$	$\boldsymbol{e}^i \otimes \boldsymbol{e}_j$

TABLE 2. Common vector identities involving the tensor product, written in both symbolic and index notation.

Name	Symbolic	Symbolic	Index
Tensor trace	$\operatorname{tr}(\boldsymbol{a} \otimes \boldsymbol{b})$	$= \boldsymbol{a} \cdot \boldsymbol{b}$	$= a_i b^i = a^i b_i$
Right operation	$(\boldsymbol{a} \otimes \boldsymbol{b})\, \boldsymbol{w}$	$= \boldsymbol{a}\,(\boldsymbol{b} \cdot \boldsymbol{w})$	$= \left(a^i b_j w^j\right) \boldsymbol{e}_i$
Left operation	$\boldsymbol{w}\,(\boldsymbol{a} \otimes \boldsymbol{b})$	$= (\boldsymbol{w} \cdot \boldsymbol{a})\, \boldsymbol{b}$	$= \left(w^j a_j b^i\right) \boldsymbol{e}_i$
Transpose	$(\boldsymbol{a} \otimes \boldsymbol{b})^T$	$= (\boldsymbol{b} \otimes \boldsymbol{a})$	$= \left(a_i b^j\right) \boldsymbol{e}_j \otimes \boldsymbol{e}^i$
Advection	$\boldsymbol{u}\,(\boldsymbol{\nabla} \otimes \boldsymbol{w})$	$= (\boldsymbol{u} \cdot \boldsymbol{\nabla})\, \boldsymbol{w}$	$= \left(u^j \nabla_j w^i\right) \boldsymbol{e}_i$
—	$\boldsymbol{u}\,(\boldsymbol{\nabla} \otimes \boldsymbol{w})^T$	$= (\boldsymbol{\nabla} \otimes \boldsymbol{w})\, \boldsymbol{u}$	$= \left(u_j \nabla_i w^j\right) \boldsymbol{e}^i$

Evaluation of the cross product and curl in index notation requires the introduction of an appropriate tensor operator. The *Levi-Civita symbol* ϵ_{ijk}, given by the rule

$$\epsilon_{ijk} = \epsilon^{ijk} \equiv \begin{cases} 1, & ijk \text{ an even permutation of } 123 \\ -1, & ijk \text{ an odd permutation of } 123 \\ 0, & \text{otherwise} \end{cases} \tag{16}$$

is used in turn to define the components of the *Levi-Civita tensor*, also known as the *alternating tensor*. Its covariant representation is formed as

$$\varepsilon_{ijk} \equiv \sqrt{g}\epsilon_{ijk}\ , \tag{17}$$

where $g \equiv |g_{ij}|$ is the determinant of the matrix composed of the metric coefficients. With orthonormal basis vectors we have $g = 1$, so in the familiar case of a Cartesian coordinate system the Levi-Civita symbol and tensor are equal. As the components of a tensor, the Levi-Civita's tensor indices may be raised in the usual way, with

$$\varepsilon^{ijk} = \frac{1}{\sqrt{g}} \epsilon^{ijk} \tag{18}$$

constituting its contravariant representation. The product of these two representations satisfies the useful identity

$$\varepsilon_{ijk} \varepsilon^{k\ell m} = \delta_i^\ell \delta_j^m - \delta_i^m \delta_j^\ell \tag{19}$$

which can be used to easily prove most of the triple-product identities that must be memorized or looked up for manipulation of symbolic equations.

TABLE 3. Operations for isolating the portions of a second-order tensor $\boldsymbol{A}$ exhibiting various types of symmetries in symbols and index notation. In the index notation expressions, the associated basis is $\boldsymbol{e}^i \otimes \boldsymbol{e}^j$ in each case.

Part	Symbolic	Index
Isotropic	$\frac{1}{3}\operatorname{tr}(\boldsymbol{A})\boldsymbol{I}$	$\frac{1}{3} g_{ij} A^k{}_k$
Symmetric	$\frac{1}{2}\left(\boldsymbol{A} + \boldsymbol{A}^T\right)$	$\frac{1}{2}\left(A_{ij} + A_{ji}\right) \equiv A_{(ij)}$
Skew-symmetric	$\frac{1}{2}\left(\boldsymbol{A} - \boldsymbol{A}^T\right)$	$\frac{1}{2}\left(A_{ij} - A_{ji}\right) \equiv A_{[ij]}$

Notation will be needed for isolating portions of a second-order tensor $\boldsymbol{A}$ exhibiting various types of symmetries, presented in Table 3. In symbolic notation, a tensor is said to *symmetric* if $\boldsymbol{A} = \boldsymbol{A}^T$ and *antisymmetric* or *skew-symmetric* if $\boldsymbol{A} = -\boldsymbol{A}^T$. In index notation, a tensor is symmetric if

$$A_{ij} = A_{ji}, \qquad A^i{}_j = A_j{}^i, \qquad A^{ij} = A^{ji} \tag{20}$$

and skew symmetric if

$$A_{ij} = -A_{ji}, \qquad A^i{}_j = -A_j{}^i, \qquad A^{ij} = -A^{ji}. \tag{21}$$

Using index juggling, all statements in (20) and (21) are shown to be true if any one of them is true. Note that for a symmetric tensor $\boldsymbol{A}$, there is no ambiguity in writing the mixed components as $A^i_j = A^i{}_j = A_j{}^i$, without the placeholder spaces; this is the case for the Kronecker delta δ^i_j.

This discussion has shown that explicitly writing the basis vectors is superfluous because all relevant information is already present in the placement of indices. However, rather than identify w^i and $A^i{}_j$ with a vector and second-order tensor respectively, as is sometimes done, we find it helpful to emphasize that w^i and $A^i{}_j$ are tensor *components* which only when paired with basis vectors yield the actual tensors. This underscores, for example, the fact that derivatives of the tensors involve derivatives of both the tensor components *and* the basis vectors, even if the basis vectors are not explicitly written.

3. Spatial derivatives in curvilinear coordinate systems

We would like to express not only changes in tensor representations but also spatial and temporal differentiation in component form without explicit reference to the basis vectors. This requires that a derivative be expressed in such a way that it involves an action only on the components of the tensor and not on the basis vectors. It can be shown that partial derivatives of tensor components such as $\partial_i w^j$ and $\partial_i A^j{}_k$ do not transform according to the tensor transformation law (25) and are thus not themselves the components of tensors. Some extension of partial derivatives is thus required to express the spatial derivatives of tensors in component form.

In section 2a and Figure 1 we introduced the deforming, accelerating coordinate system x^i—the typical setting for GFD—as well as the rigid, inertial coordinate system y^a. As discussed therein, these have respective basis vectors $\boldsymbol{e}_i$ and $\boldsymbol{e}_a$ and position vectors $\boldsymbol{r}$ and $\boldsymbol{s}$. Quantities associated with these two systems are distinguished by the use of the index sets $\{i, j, k, \ell, m\}$ and $\{a, b, c, d, e\}$ respectively; for example $\boldsymbol{r} = r^i \boldsymbol{e}_i = r^a \boldsymbol{e}_a$ connects the position $\boldsymbol{r}$ in the x^i coordinate system to its representation both the x^i and the y^a systems. We will now examine how to represent any tensors consistently in both of these coordinate systems.

The Christoffel symbols defined in (5) can be expressed in terms of the metric coefficients as

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{i\ell} (\partial_j g_{\ell k} + \partial_k g_{\ell j} - \partial_\ell g_{jk}) \; . \qquad (22)$$

Essentially all information that is required within a given coordinate system is found by computing the basis vectors via (1), the metric coefficients in terms of the basis vectors via (10), and the Christoffel symbols in terms of the metric coefficients via (22).

Additional information is required to change tensors between two coordinate representations. This is accomplished via the *Jacobian coefficients*

$$J_i^a \equiv \partial_i y^a, \qquad J_a^i \equiv \partial_a x^i \qquad (23)$$

which are readily verified to be given in terms of the basis vectors by[1]

$$J_i^a = \boldsymbol{e}_i \cdot \boldsymbol{e}^a, \qquad J_a^i = \boldsymbol{e}_a \cdot \boldsymbol{e}^i \; . \qquad (24)$$

Since $\boldsymbol{e}_i = J_i^a \boldsymbol{e}_a$, the J_i^a are the components of the $\boldsymbol{e}_i$ basis expanded in terms of the $\boldsymbol{e}_a$ basis, and conversely for J_a^i.

[1]To see this, note that the Jacobian J_a^i may be used to transform the basis vectors as $\boldsymbol{e}_a \equiv \partial_a \boldsymbol{s} = \partial_a \boldsymbol{r} = J_a^i \partial_i \boldsymbol{r} = J_a^i \boldsymbol{e}_i$. It then follows that $J_i^a = J_j^a \delta_i^j = J_j^a (\boldsymbol{e}_i \cdot \boldsymbol{e}^j) = \boldsymbol{e}_i \cdot J_j^a \boldsymbol{e}^j = \boldsymbol{e}_i \cdot \boldsymbol{e}^a$, and similarly for J_a^i.

Jacobian coefficients may be combined so as to contract the indices of a tensor of any order and valence, e.g.,

$$w^a = J_i^a w^i, \qquad w_a = J_a^i w_i, \qquad A^a{}_b = J_i^a J_b^j A^i{}_j, \qquad (25)$$

a pattern known as the *tensor transformation law*. An indexed entity contains the components of a tensor *if and only if* it transforms in this way. Indeed, this transformation law can be used as an alternate, equivalent definition of a tensor to the definition of a tensor as a multilinear function on vectors, as shown in (1.6)–(1.10) of Jeevanjee (2011).

The Christoffel symbols $\Gamma^k{}_{ij}$ do not transform according to the tensor transformation law and are therefore not the components of a third-order tensor. A tensor with the same mixture of indices as $\Gamma^k{}_{ij}$ would transform according to

$$A^a{}_{bc} = J_i^a J_b^j J_c^k A^i{}_{jk} \; . \qquad (26)$$

The Christoffel symbols, however, follow the transformation law

$$\Gamma^a{}_{bc} = J_i^a J_b^j J_c^k \Gamma^i{}_{jk} + \boxed{J_i^a \frac{\partial^2 x^i}{\partial y^b \partial y^c}} \qquad (27)$$

which differs from (26) due to the boxed term. The reason the Christoffel symbols are not components of tensors is that $\Gamma^k{}_{ij}$, by construction according to (5), describes an aspect of the x^i coordinate system, namely how its basis vectors vary spatially. When we change coordinates to y^a, we are not simply changing the representation of the Christoffel symbols $\Gamma^k{}_{ij}$ but rather obtaining new quantities $\Gamma^a{}_{bc}$ that describe the spatial variations of a different set of basis vectors, the basis vectors $\boldsymbol{e}_a$ of the y^a system.

Nevertheless, the Christoffel symbols can be said to resemble tensors in the sense that the upper index and/or either one of the two lower indices will transform according to the transformation law, but both lower indices together will not. This can be understood in terms of their definitions $\Gamma^k{}_{ij} \equiv (\partial_i \boldsymbol{e}_j) \cdot \boldsymbol{e}^k$ and $\Gamma^c{}_{ab} \equiv (\partial_a \boldsymbol{e}_b) \cdot \boldsymbol{e}^c$. The upper index k of $\Gamma^k{}_{ij}$ corresponds to the basis vectors we are expanding in terms of; this index will therefore transform like the contravariant components of a tensor. The first lower index i corresponds to the coordinate system in which we are taking a partial derivative; this index will transform like the covariant components of a tensor. From the symmetry $\Gamma^k{}_{ij} = \Gamma^k{}_{ji}$, the second lower index can also be transformed like a tensor. Yet $\partial_i \boldsymbol{e}_j$ together does not transform like a tensor in j because this would require $\partial_i J_j^a = 0$, which is not the case in general; thus both lower indices do not simultaneously transform according to the tensor transformation law.

a. Changing representations involve the identity tensor

It builds intuition to highlight a detail that we have not found to be discussed in the literature, namely the role of the identity tensor $\boldsymbol{I}$ and its relatives in changing representations. As discussed in section 2d, in Euclidean space the action of index juggling implicitly involves the identity tensor, since $g^{ij}w_j = w^i$ corresponds to the tensor equation $\boldsymbol{I}\boldsymbol{w} = \boldsymbol{w}$ with $\boldsymbol{I} = g^{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$. However, the identity tensor can also be written in the form

$$\boldsymbol{I} = J_i^a \boldsymbol{e}_a \otimes \boldsymbol{e}^i = J_a^i \boldsymbol{e}_i \otimes \boldsymbol{e}^a, \tag{28}$$

in which case the coefficients of the basis vectors are seen to be the Jacobian coefficients. In this form it is clear that changing representations from one basis to another also involves the identity tensor, since $w^a = J_i^a w^i$ again corresponds to $\boldsymbol{I}\boldsymbol{w} = \boldsymbol{w}$ but now with $\boldsymbol{I} = J_i^a \boldsymbol{e}_a \otimes \boldsymbol{e}^i$. In both cases the effect is to express the tensor in a new representation without changing the tensor itself. Moreover, combining (15) with (28), we also see that in Euclidean space, the Kronecker delta, the metric coefficients, and the Jacobian coefficients are all the components of the identity tensor $\boldsymbol{I}$ with respect to different representations.

This clarifies several points of ambiguity in the literature. Commonly g_{ij} is referred to as the *metric tensor*, but there is no separate "metric" tensor having the g_{ij} as its covariant components apart from the identity tensor $\boldsymbol{I} = g_{ij}\boldsymbol{e}^i \otimes \boldsymbol{e}^j$. For this reason we refer to g_{ij} as the metric *coefficients* as opposed to the metric *tensor*. At the same time, is also often said that the Jacobians J_i^a and J_a^i are not tensors. However, in Euclidean space the Jacobian coefficients have the same status as the metric coefficients—they are components of the identity tensor, but in a representation that uses a basis and a reciprocal basis drawn from two different coordinate system, as in (28). The equal status of these two sets of coefficients is also apparent if one observes that $g_{ij} \equiv \boldsymbol{e}_i \cdot \boldsymbol{e}_j$ while similarly $J_a^i = \boldsymbol{e}_a \cdot \boldsymbol{e}^i$.

b. Spatial differentiation

Taking the partial spatial derivative of a vector $\boldsymbol{w}$ with respect to the ith coordinate, one finds

$$\partial_i \boldsymbol{w} = \partial_i (w^j \boldsymbol{e}_j) = \boldsymbol{e}_j \partial_i w^j + \boxed{w^j \partial_i \boldsymbol{e}_j}\,, \tag{29}$$

which illustrates that the components of the derivative of a tensor are not the derivatives of its components; spatial variations in the basis vectors will contribute via the boxed term. The partial derivative $\partial_i \boldsymbol{w}$ can be rewritten as

$$\partial_i \boldsymbol{w} = \boldsymbol{e}_j \nabla_i w^j, \qquad \nabla_i w^j \equiv \partial_i w^j + \Gamma^j{}_{ik} w^k \tag{30}$$

using the property of the Christoffel symbol in (4). Here $\nabla_i w^j$ is known as the *covariant derivative* of w^j, and has been constructed to express the action of taking the derivative of a vector in terms of an operation on its components.

The covariant derivative of the components of a tensor will take on a different form depending upon the tensor's order and the valence of its representation. For a scalar or zero-rank tensor field we have the simple relation $\nabla_i a \equiv \partial_i a$, while for a second-order tensor $\boldsymbol{A}$ expressed in mixed component form the covariant derivative is

$$\nabla_i A_k^j \equiv \partial_i A_k^j + \Gamma^j{}_{i\ell} A_k^\ell - \Gamma^\ell{}_{ik} A_\ell^j. \tag{31}$$

This pattern extends in the obvious way to tensor components of different orders and valences, with a positive Christoffel term for each upper index and a negative term for each lower index. Importantly, the covariant derivative exhibits the usual properties, such as the product rule,

$$\nabla_i \left(a^j b_k\right) = a^j \nabla_i b_k + b_k \nabla_i a^j, \tag{32}$$

and the chain rule, behaviors which arise from the fact that its form changes depending upon the valence of its operand, see section 8.6 of Grinfeld (2013).

It can be shown that $\nabla_i w^j$ and $\nabla_i A^j{}_k$, unlike the corresponding partial derivatives $\partial_i(\cdot)$, transform according to the tensor transformation law (25) when w^j and $A^j{}_k$ do, and they are consequently the components of a second and third-order tensor, respectively. This means we can raise the index of a covariant derivative in the usual way via

$$\nabla^i \equiv g^{ij}\nabla_j\,, \tag{33}$$

a construction that is called the *contravariant derivative*.

In Table 1 we see how the covariant derivative is used to various types of spatial derivatives as actions on tensor components. In general, the application of the covariant derivative to a tensor of order k yields an order $k+1$ tensor. Importantly, when the coordinate system spans the three dimensions of Euclidean space, covariant differentiation of a tensor component corresponds to partial differentiation of the tensor itself, as is evident from fact that $\nabla_i w^j$ is the jth component of $\partial_i \boldsymbol{w}$ from (30); this will no longer be the case when the covariant derivative is applied along a curved manifold, such as in general relativity or along a curved surface embedded in three dimensional Euclidean space, see Chapter 11 of Grinfeld (2013).

The gradient tensor, commonly written symbolically as $\boldsymbol{\nabla}\boldsymbol{w}$, is here written in the more explicit form $\boldsymbol{\nabla} \otimes \boldsymbol{w}$, for reasons discussed in section 3e of Lilly et al. (2024). A notable gradient tensor is the gradient of the position vector $\boldsymbol{r}$, for which one finds

$$\boldsymbol{\nabla} \otimes \boldsymbol{r} = \boldsymbol{I}, \qquad \nabla_j r^i = \delta_j^i\,, \tag{34}$$

since $\boldsymbol{\nabla} \otimes \boldsymbol{r} = \boldsymbol{e}^i \otimes \partial_i \boldsymbol{r} = \boldsymbol{e}^i \otimes \boldsymbol{e}_i = \boldsymbol{I}$ from the definition $\boldsymbol{\nabla} \equiv \boldsymbol{e}^i \partial_i$ together with $\boldsymbol{e}_i \equiv \partial_i \boldsymbol{r}$ in (1). Another is the gradient tensor associated with the jth basis vector,

$$\boldsymbol{\nabla} \otimes \boldsymbol{e}_j = \boldsymbol{e}^i \otimes \partial_i \boldsymbol{e}_j = \Gamma^k{}_{ij} \left(\boldsymbol{e}^i \otimes \boldsymbol{e}_k \right), \tag{35}$$

which clearly reveals that the upper and one lower index of the Christoffel symbol transform like tensor components, in agreement with discussion in the previous subsection.

The covariant derivative has the crucial property that its application to the metric coefficients vanishes,

$$\nabla_i g^{jk} = 0, \qquad \nabla_i g_{jk} = 0, \qquad \nabla_i \delta^j_k = 0, \tag{36}$$

a result known as *Ricci's theorem*, *metric compatibility*, or the *metrinilic property*. As shown in Appendix A, its application to the Jacobian coefficients also vanishes

$$\nabla_i J^a_j = 0, \quad \nabla_i J^j_a = 0, \quad \nabla_a J^i_b = 0, \quad \nabla_a J^b_i = 0. \tag{37}$$

It may similarly be shown that the application of the covariant derivative to the Levi-Civita tensor vanishes,

$$\nabla_\ell \varepsilon_{ijk} = 0, \qquad \nabla_\ell \varepsilon^{ijk} = 0, \tag{38}$$

meaning that the covariant derivative and the Levi-Civita tensor also commute.

Properties (36) and (37) together mean that the covariant derivative commutes with changes in representation, e.g.,

$$g_{kj} \nabla_i w^j = \nabla_i \left(g_{kj} w^j \right), \qquad J^a_j \nabla_i w^j = \nabla_i \left(J^a_j w^j \right). \tag{39}$$

Thus $\nabla_i w^j$ will transform valences via the metric coefficients and will also transform basis representations according to the tensor transformation law; this behavior is required in order that $\nabla_i w^j$ be the components of the derivative of a tensor, independent of any representation. The ordinary partial derivative does not, in general, commute with changes of representation. It will be seen subsequently that many of the apparent forces of encountered in GFD, as well as new apparent forces arising in deforming coordinate systems, manifest in index notation as differences between covariant derivatives and partial derivatives of tensor components.

c. Index swapping rules

In changing representations between the $\boldsymbol{e}_i$ and the $\boldsymbol{e}_a$ bases, the following rules will prove useful. Any repeated indices that generate proper tensors may be freely swapped between the $\{i, j, k, \ell, m\}$ and $\{a, b, c, d, e\}$ index sets, e.g.,

$$w^i \boldsymbol{e}_i = w^a \boldsymbol{e}_a, \qquad A_i{}^j \boldsymbol{e}^i \otimes \boldsymbol{e}_j = A_i{}^b \boldsymbol{e}^i \otimes \boldsymbol{e}_b\ . \tag{40}$$

Any contracted indices on tensor components can likewise be swapped, e.g.,

$$u^i v_i = u^a v_a, \qquad A_{ab} w^b = A_{ai} w^i. \tag{41}$$

For the Christoffel symbols, contractions that involve the upper index and/or one of the two lower indices may be swapped, but not those involving both lower indices:

$$\begin{aligned} \Gamma^i{}_{jk} u_i v^j w^k &= \Gamma^a{}_{jk} u_a v^j w^k = \Gamma^a{}_{bk} u_a v^b w^k \\ &= \Gamma^a{}_{jc} u_a v^j w^c \neq \Gamma^a{}_{bc} u_a v^b w^c. \end{aligned} \tag{42}$$

Although partial derivatives of tensor components are not in general tensor components, indices of *contracted* partial derivatives may also be swapped via

$$u^i \partial_i v^j = u^a \partial_a v^j. \tag{43}$$

This is so because we can use (30) to rewrite expressions involving partial derivatives in terms of the covariant derivative and the Christoffel symbols, for example

$$u^i \partial_i v^j = u^i \nabla_i v^j - \Gamma^j{}_{ik} u^i v^k. \tag{44}$$

We see that swapping the i index for a only requires one of the two lower indices of the Christoffel symbol to change, and is therefore permissible.

4. Time derivatives in moving coordinate systems

In order to determine the momentum equation in a generally accelerating, rotating, and deforming coordinate system, we also need an analogue of the covariant derivative for temporal differentiation. That is, we need a way to compute total time derivatives of tensor fields by operating only on their components. This section introduces such a derivative, first presenting a known result for inertial coordinate systems, and then introducing a generalization that accommodates moving, deforming coordinates.

a. Frames and settings

Consider the coordinate system y^a, which we have introduced as being both rigid and inertial. In fact, both of these properties more properly pertain not to the coordinate system y^a, but rather to the *frame* in which it is embedded. A frame is understood as an imaginary, by definition rigid lattice extending throughout space, in comparison to which Newtonian motion may be measured (see e.g., Koks 2017). Given a non-accelerating frame, any curvilinear coordinate system that is fixed with respect to that frame will inherit its rigidity and inertiality. Thus, referring to y^a as a rigid, inertial *coordinate system* is simply a shorthand for saying that it is a coordinate system embedded within an inertial *frame*. Recall that we have employed Galilean invariance to regard this inertial frame as stationary.
f

The x^i coordinates surfaces, by contrast, are allowed to evolve with respect to the inertial frame according to any combination of translation, rotation, and deformation.

TABLE 4. A table of different time derivatives appearing in this work. χ is a general path along which differentiation occurs. We understand the material derivative of a tensor to be taken following a fixed parcel label ξ. The intrinsic derivatives are constructed to produce components of the corresponding total derivatives acting on tensors. One further sort of time derivative, the convected derivative of Oldroyd (1950), is discussed only in Appendix F and omitted here.

Name	Symbol	Argument
Total along path χ	$\frac{d}{dt}\big\rvert^{\chi}$	Tensor
Total Material	$\frac{D}{Dt} \equiv \frac{d}{dt} \equiv \frac{d}{dt}\big\rvert^{\chi_\xi} \equiv \frac{d}{dt}\big\rvert_{\xi}$	Tensor
Partial with fixed x	$\frac{\partial}{\partial t}\big\rvert_{x}$	Either
Partial with fixed y	$\frac{\partial}{\partial t}\big\rvert_{y}$	Either
Covariant Advection	$\nabla_{\boldsymbol{v}} \equiv v^k \nabla_k = v^c \nabla_c$	Components
Lie Advection	$\mathcal{L}_{\boldsymbol{v}}$	Components
Intrinsic	$\frac{\delta}{\delta t}\big\rvert^{\chi}$	Components
Intrinsic Material	$\frac{\delta}{\delta t} \equiv \frac{\delta}{\delta t}\big\rvert^{\chi_\xi} \equiv \frac{\delta}{\delta t}\big\rvert_{\xi}$	Components

Because these coordinates are permitted to exhibit non-rigid motion, it is not possible to fix them within any rigid frame. We instead introduce the term *setting* to indicate a deformable frame-like entity, visualizable in two dimensions as an evolving, stretching rubber sheet onto which coordinate lines are marked; in the case without deformation, a setting becomes a frame. For brevity we refer to the x^i coordinates as being within the *moving setting*. In what follows, we will relate motion with respect to the moving setting to motion with respect to the inertial frame in order to formulate dynamical laws in the moving setting.

Since the x^i coordinate surfaces are allowed to be temporally varying, it follows that the basis vectors $\boldsymbol{e}_i$, metric coefficients g_{ij} and g^{ij}, and Christoffel symbols $\Gamma^k{}_{ij}$ are as well. Meanwhile, the y^a coordinate surfaces and hence basis vectors $\boldsymbol{e}_a$, metric coefficients g_{ab} and g^{ab}, and Christoffel symbols $\Gamma^c{}_{ab}$ are all temporally constant. The Jacobian coefficients J^a_i and J^i_a for converting between the two coordinate systems will also vary in time as the x^i coordinates do. All of these quantities are permitted to be spatially varying.

The two coordinate systems are related as follows, as sketched in Fig. 1. The origin $O_x(t)$ of the coordinate system in the moving setting is located at position vector $\boldsymbol{S}(t)$ with respect to the origin O_y of the coordinate system in the inertial frame. Let $\boldsymbol{r}$ be a position vector with respect to the origin of the moving coordinate system O_x, and $\boldsymbol{s}$ a position vector with respect to the origin of the inertial coordinate system O_y, with $\boldsymbol{s} = \boldsymbol{r} + \boldsymbol{S}$.

It is assumed that our coordinate systems are sufficiently well-behaved that they can be related by invertible sets of differentiable functions, allowing one to translate freely between the names given to a particular point in each system. That is, we can write each coordinate system in terms of the other as

$$x^i(y,t), \qquad y^a(x,t), \tag{45}$$

where we suppress coordinate indices within function arguments. Differentiating each set of coordinates with the other held fixed yields two velocities fields, $\mathcal{U}$ and $\mathcal{V}$, having components

$$\mathcal{U}^i \equiv \left.\frac{\partial x^i}{\partial t}\right|_y, \qquad \mathcal{V}^a \equiv \left.\frac{\partial y^a}{\partial t}\right|_x. \tag{46}$$

At each point in space $\mathcal{U} = \mathcal{U}^i \boldsymbol{e}_i$ is the velocity of the inertial frame as observed from the moving setting, while $\mathcal{V} = \mathcal{V}^a \boldsymbol{e}_a$ is the velocity of the moving setting as observed from the inertial frame.

Clearly these velocity fields are equal and opposite, $\mathcal{U} = -\mathcal{V}$, although they both may vary in space and time. Following the tensor transformation law, each field is expressible in terms of the other basis as

$$\mathcal{U}^a \equiv J^a_i \mathcal{U}^i, \qquad \mathcal{V}^i \equiv J^i_a \mathcal{V}^a, \tag{47}$$

where $J^a_i \equiv \partial_i y^a$ and $J^i_a \equiv \partial_a x^i$ are the Jacobian coefficients defined earlier in (23), such that $\mathcal{V} = \mathcal{V}^i \boldsymbol{e}_i = \mathcal{V}^a \boldsymbol{e}_a$ and similarly for $\mathcal{U}$. Due to the fact that the inertial coordinate system is assumed to be stationary, $\mathcal{V}$ constitutes the absolute velocity of the moving setting. Hereafter we use only $\mathcal{V}$, which we term the *setting velocity field*.

Let the scalar field ξ be a label that remains constant following fluid parcels. Differentiating each set of coordinates with the label ξ held fixed yields two velocity fields characterizing the fluid motion, $\boldsymbol{u} = u^i \boldsymbol{e}_i$ and $\boldsymbol{v} = v^a \boldsymbol{e}_a$, with components given by

$$u^i \equiv \left.\frac{dx^i}{dt}\right|_\xi, \qquad v^a \equiv \left.\frac{dy^a}{dt}\right|_\xi. \tag{48}$$

The velocity field $\boldsymbol{u}$ is the *relative velocity* observed with respect to the moving setting while $\boldsymbol{v}$ is the *absolute velocity*. Although these velocity fields make the most sense in the setting where they are easily observed, they are tensor fields and can therefore be expressed in components relative to the basis vectors of the other setting,

$$u^a \equiv J^a_i u^i, \qquad v^i \equiv J^i_a v^a. \tag{49}$$

These two velocity fields are related as follows. Considering the x^i coordinates to be a function of parcel label and time, $x^i(\xi,t)$, we have

$$\frac{d}{dt}y^a[x(\xi,t),t]\bigg|_\xi = J^a_i \frac{dx^i}{dt}\bigg|_\xi + \frac{\partial y^a}{\partial t}\bigg|_x, \tag{50}$$

which reduces to $v^a = u^a + \mathcal{V}^a$. Consequently

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{\mathcal{V}}. \tag{51}$$

The absolute velocity of the fluid $\boldsymbol{v}$ is thus the sum of its velocity $\boldsymbol{u}$ relative to the moving setting and the setting velocity $\boldsymbol{\mathcal{V}}$ itself.

In typical GFD configurations in which the coordinate system x^i is embedded within a frame of reference fixed with respect to a rotating planet, $\boldsymbol{\mathcal{V}}$ would correspond to the planetary velocity, and $\boldsymbol{u}$ to the velocity relative to the planet. In cases where the x^i coordinates deform relative to the planet, e.g., in isopycnal or pressure coordinates, the setting velocity $\boldsymbol{\mathcal{V}}$ would capture this coordinate motion as well as the planetary motion.

b. The intrinsic derivative in static coordinates

Next we present a standard indicial expression for the derivative along an arbitrary curve, the *intrinsic derivative*, which allows us to express the material derivative following fluid parcels as a special case. The intrinsic derivative is defined in the literature only for *static* coordinate systems, by which we mean a coordinate system whose basis vectors do not change as a function of time. This is not quite the same as an inertial coordinate system, because a coordinate system embedded in a linearly accelerating frame is static but not inertial; an inertial system is therefore static but the converse is not necessarily true. This form of the intrinsic derivative thus requires generalization to accommodate rotating or deforming coordinate systems, such as the x^i coordinates.

Let $\chi(t)$ be some parametrized curve, with coordinates $\chi^a(t)$ and $\chi^i(t)$ within the y^a and x^i coordinate systems respectively, and write $f[\chi(t),t]$ to denote the restriction of the field f to the curve χ. We use the notation

$$\frac{df}{dt}\bigg|^\chi \equiv \frac{d}{dt}f[\chi(t),t] \tag{52}$$

to mean the total time derivative of f evaluated along χ, a quantity that is is independent of the coordinates we use to represent χ. This notation is intended to be distinct from $df/dt|_Q$, which means that the derivative is taken with the quantity Q held at a fixed value.

The rates of change of the x^i and y^a coordinates as one moves along the curve χ specify two sets of tangent vectors, $\boldsymbol{p}[\chi(t)] = p^i \boldsymbol{e}_i$ and $\boldsymbol{q}[\chi(t)] = q^a \boldsymbol{e}_a$, called the *parametric velocities*, the components of which are defined as

$$p^i \equiv \frac{dx^i}{dt}\bigg|^\chi, \qquad q^a \equiv \frac{dy^a}{dt}\bigg|^\chi. \tag{53}$$

These give the apparent velocity of motion following the parameterized curve as observed with respect to the moving setting and the inertial frame, respectively. We emphasize that these are *not* material velocities—they correspond to the rate of change of a position along any chosen curve $\chi(t)$, and are in general completely independent of the velocity of the fluid in which the curve is embedded.

When f is considered to be a function of the y^a coordinates, $f = f[\chi^a(t),t]$, its derivative along χ is

$$\frac{df}{dt}\bigg|^\chi = \frac{\partial f}{\partial t}\bigg|_y + q^a \partial_a f \tag{54}$$

from the chain rule. When applied to a vector field $\boldsymbol{w}$, the total derivative along χ can be expressed as

$$\frac{d\boldsymbol{w}}{dt}\bigg|^\chi = \frac{\partial w^a}{\partial t}\bigg|_y \boldsymbol{e}_a + \left(q^b \nabla_b w^a\right)\boldsymbol{e}_a \tag{55}$$

from the fact that the basis vectors $\boldsymbol{e}_a$ are temporally constant combined with the definition of the covariant derivative in (30). We can thus define the *intrinsic derivative*

$$\frac{\delta}{\delta t}\bigg|^\chi \equiv \left(\frac{\partial}{\partial t}\bigg|_y + q^b \nabla_b\right), \tag{56}$$

acting on tensor components, allowing us to write

$$\frac{d\boldsymbol{w}}{dt}\bigg|^\chi = \frac{\delta w^a}{\delta t}\bigg|^\chi \boldsymbol{e}_a. \tag{57}$$

See §7.55 of Aris (1962) or §8.9 of Grinfeld (2013) for extended discussion of this operator.

The action of the intrinsic derivative is to express the total time derivative along the curve $\chi(t)$ in static but curvilinear coordinate systems in terms of operations on tensor components alone. In other words, when we apply the intrinsic derivative to the components of a tensor, what we are doing is taking the total time derivative of that tensor along the specified curve. The form of the intrinsic derivative given in (56) is independent of the order and valence of the tensor components on which it operates because this dependence is already accounted for by the covariant derivative. However, (56) is only valid for tensor components expressed in coordinate systems in which the basis vectors do not change as a function of time.

c. The material derivative in static coordinates

We now choose a curve, denoted $\chi_\xi(t)$, to follow the fluid parcel ξ, and then allow ξ to take on all possible label values, such that $\chi_\xi(t)$ specifies the paths of all fluid parcels. The total derivative along the curve $\chi_\xi(t)$ following fluid parcels for each value of ξ—that is, the *material derivative*—of some tensor quantity f is then given by

$$\left.\frac{df}{dt}\right|^{\chi_\xi} = \left.\frac{df}{dt}\right|_\xi = \frac{d}{dt} f[\chi_\xi(t), t]. \tag{58}$$

From (53), the parametric velocities components p^i and q^a evaluated along the paths of fluid parcels become

$$p^i|_\xi \equiv \left.\frac{dx^i}{dt}\right|^{\chi_\xi} = u^i, \qquad q^a|_\xi \equiv \left.\frac{dy^a}{dt}\right|^{\chi_\xi} = v_a. \tag{59}$$

as follows by comparison with (48). This emphasizes that the fluid velocities $\boldsymbol{u}$ and $\boldsymbol{v}$ are a special case of the parametric velocities $\boldsymbol{p}$ and $\boldsymbol{q}$ along a family of curves, when those curves are taken to follow fluid parcels.

Applied to a vector field $\boldsymbol{w}$, the material derivative can be written in terms of the inertial y^a coordinates as

$$\left.\frac{d\boldsymbol{w}}{dt}\right|^{\chi_\xi} = \frac{\delta w^a}{\delta t}\boldsymbol{e}_a, \qquad \frac{\delta}{\delta t} \equiv \left.\frac{\partial}{\partial t}\right|_y + v^b \nabla_b \tag{60}$$

where the action of the material derivative on tensor components is given by the *material intrinsic derivative* operator $\delta/\delta t$ (Aris 1962; Grinfeld 2013). This is simply the intrinsic derivative (56) evaluated along fluid parcel trajectories, for which the parametric velocity $\boldsymbol{q}$ has been set to the absolute fluid velocity $\boldsymbol{v}$. Again, the form (60) is only valid for static coordinate systems. Note that we have defined the symbol $\delta/\delta t$ to a be special case of the intrinsic derivative along a curve. In the absence of the specification of a particular curve through the "$(\cdot)|^\chi$" notation, it is understood that this derivative is to be evaluated along the set of curves comprising all fluid parcel trajectories.

d. Velocities and settings

Before proceeding, it is important to gain further insight into the relationship between the vector fields $\boldsymbol{p}$ and $\boldsymbol{q}$, which will also help clarify the notion of a setting. As in Fig. 1, let $\boldsymbol{r}$ and $\boldsymbol{s}$ be the position vectors measured from the origins O_x and O_y respectively to some point in space. Due to the motion of the coordinate system, $\boldsymbol{r}$ may change even with coordinates x^i held fixed. However, on account of our choice that the inertial coordinate system y^a is regarded to be embedded within a stationary frame, $\boldsymbol{s}$ may not change with y^a held fixed. Consequently, considering the position vectors along the curve $\chi(t)$, we see that $\boldsymbol{r}[\chi(t), t]$ is an explicit function of time but $\boldsymbol{s}[\chi(t)]$ is not.

Taking the time derivatives of these position vector along the curve $\chi(t)$ then leads to

$$\left.\frac{d\boldsymbol{r}}{dt}\right|^{\chi} = \boldsymbol{p} + \left.\frac{\partial \boldsymbol{r}}{\partial t}\right|_x, \qquad \left.\frac{d\boldsymbol{s}}{dt}\right|^{\chi} = \boldsymbol{q} \tag{61}$$

from $\boldsymbol{e}_i \equiv \partial_i \boldsymbol{r}$ and $\boldsymbol{e}_a \equiv \partial_a \boldsymbol{s}$ together with the definitions of $\boldsymbol{p}$ and $\boldsymbol{q}$ in (53). Thus the rate of change of the position vector $\boldsymbol{s}$ measured with respect the inertial frame (which has been chosen to be stationary) is the same as the vector field $\boldsymbol{q}$ observed in that frame, but the same is not true for the moving setting. In the moving setting, as additional contribution $\partial \boldsymbol{r}/\partial t|_x$ appears due to the motion of the coordinate surfaces themselves. We term $\partial \boldsymbol{r}/\partial t|_x$ the *inherent velocity field* since it is an inherent property of the setting, not of the fluid motion. The simplest case of a non-zero inherent velocity field is that due to rigid rotation.

We may now make the following key observation with respect to settings: if two different coordinate systems have the same inherent velocity fields at all times and locations, they share the same setting. In this case, all coordinate locations in one coordinate system remain at the same coordinate locations in the other system even as the setting moves or deforms. These two coordinate systems can then be visualized in two dimensions as two different sets of lines on the same deformable rubber sheet. Like a frame, a setting therefore does not depend upon or imply a choice of coordinates, even if a particular coordinate system was involved in its initial specification.

The inherent velocity field $\partial \boldsymbol{r}/\partial t|_x$ is related to the setting velocity $\boldsymbol{\mathcal{V}}$, as will now be shown. Since $\boldsymbol{s} = \boldsymbol{r} + \boldsymbol{S}$, see Fig. 1, we can take the total derivative of this equation along material curves $\chi_\xi(t)$ to yield, using (61) and (59),

$$\boldsymbol{v} = \boldsymbol{u} + \left.\frac{\partial \boldsymbol{r}}{\partial t}\right|_x + \frac{d\boldsymbol{S}}{dt}. \tag{62}$$

Since $\boldsymbol{S}$ is spatially constant for a given time, we also have

$$\frac{d\boldsymbol{S}}{dt} = \left.\frac{\partial \boldsymbol{S}}{\partial t}\right|_x = \left.\frac{\partial \boldsymbol{S}}{\partial t}\right|_y \tag{63}$$

Comparison with (51) then shows

$$\boldsymbol{\mathcal{V}} = \left.\frac{\partial \boldsymbol{r}}{\partial t}\right|_x + \frac{d\boldsymbol{S}}{dt} = \left.\frac{\partial \boldsymbol{s}}{\partial t}\right|_x, \tag{64}$$

which means that the setting velocity field $\boldsymbol{\mathcal{V}}$ consists of the inherent velocity of the moving setting $\partial \boldsymbol{r}/\partial t|_x$ plus $d\boldsymbol{S}/dt$, the velocity of the origin of the moving setting. This is equivalent to the apparent velocity of the absolute location vector $\boldsymbol{s}$ as viewed from the moving setting.

e. Transforming rates of change

In order to derive the intrinsic derivative in the moving setting we must be able to transform temporal derivatives between the inertial frames and the moving setting. The partial time derivative of a function f at a fixed x location is, by the chain rule,

$$\left.\frac{\partial f}{\partial t}\right|_x = \left.\frac{\partial f}{\partial t}\right|_y + \mathcal{V}^a \partial_a f \tag{65}$$

where f is regarded first as a function of the x^i and then of the y^a coordinates. Thus

$$\left.\frac{\partial}{\partial t}\right|_x = \left(\left.\frac{\partial}{\partial t}\right|_y + \mathcal{V}^b \partial_b \right) \tag{66}$$

where the operand field on the left-hand side is understood to depend explicitly on x while that on the right-hand side is understood to depend explicitly on y, as indicated respectively by the "$|_x$" and "$|_y$" notation. Proceeding as above but for the derivative $\partial f/\partial t|_y$, one finds

$$\left.\frac{\partial}{\partial t}\right|_y = \left(\left.\frac{\partial}{\partial t}\right|_x - \mathcal{V}^j \partial_j \right) \tag{67}$$

as the inverse transformation to (66). These relationships are a consequence only of functional dependence and therefore hold for general functions, which might be tensors of any order, their components, or even non-tensor functions like the Christoffel symbols.

While the rates of change of the basis vectors $\boldsymbol{e}_a$ of the inertial frame vanish, those of the basis vectors $\boldsymbol{e}_i$ of the moving setting at a fixed x location are found to be

$$\left.\frac{\partial \boldsymbol{e}_i}{\partial t}\right|_x = \boldsymbol{e}_j \nabla_i \mathcal{V}^j \ . \tag{68}$$

Temporal changes in $\boldsymbol{e}_i$ and spatial gradients of $\boldsymbol{\mathcal{V}}$ are then just two representations of the same phenomenon.[2] We may also note the useful fact that the temporal derivative of the setting velocity $\boldsymbol{\mathcal{V}}$ at a fixed location in the inertial frame can be expressed as

$$\left.\frac{\partial \boldsymbol{\mathcal{V}}}{\partial t}\right|_y = \left.\frac{\partial \mathcal{V}^a}{\partial t}\right|_y \boldsymbol{e}_a = \left.\frac{\partial \mathcal{V}^i}{\partial t}\right|_x \boldsymbol{e}_i . \tag{69}$$

Thus the components of the time derivative $\partial\boldsymbol{\mathcal{V}}/\partial t|_y$ of the setting velocity with respect to the $\boldsymbol{e}_a$ or $\boldsymbol{e}_i$ basis are found by differentiating the components of $\boldsymbol{\mathcal{V}}$ in that basis with the corresponding coordinate set held fixed. This property is particular to the setting velocity $\boldsymbol{\mathcal{V}}$ and is not true for tensors in general.[3]

f. The intrinsic derivative in moving settings

We are now able to generalize the intrinsic derivative to a rotating, deforming setting. The derivative of vector field $\boldsymbol{w}$ along a curve χ, as expressed in terms its contravariant components in the moving coordinate system x^i, is

$$\begin{aligned} \left.\frac{d\boldsymbol{w}}{dt}\right|^{\chi} &= \left(\left.\frac{\partial}{\partial t}\right|_x + p^j \partial_j \right) \left(w^i \boldsymbol{e}_i \right) \\ &= \left.\frac{\partial w^i}{\partial t}\right|_x \boldsymbol{e}_i + p^j \nabla_j \boldsymbol{w} + \boxed{ w^i \left.\frac{\partial \boldsymbol{e}_i}{\partial t}\right|_x } \end{aligned} \tag{70}$$

using the definition of p^i in (53). The boxed term is a new sort of term not occurring in a static coordinate system, but common in GFD settings. This term gives the contribution to the rate of change of $\boldsymbol{w}$ along $\chi(t)$ arising from the temporal variability of the basis vectors.

The intrinsic derivative of the contravariant components of a vector in a moving coordinate system is then defined as

$$\left.\frac{\delta w^i}{\delta t}\right|^{\chi} \equiv \left.\frac{\partial w^i}{\partial t}\right|_x + p^j \nabla_j w^i + \boxed{ w^j \nabla_j \mathcal{V}^i } \tag{71}$$

which supersedes the earlier definition in (56). The new, boxed term emerges from the rate of change of the basis vectors in (70) on account of (68). This boxed term vanishes for any static coordinate system, recovering the form of the intrinsic derivative given earlier in (56). With this definition, we can express the total derivative along $\chi(t)$ in the moving coordinate system as

$$\left.\frac{d\boldsymbol{w}}{dt}\right|^{\chi} = \left.\frac{\delta w^i}{\delta t}\right|^{\chi} \boldsymbol{e}_i \tag{72}$$

just as was the case for the static coordinate system in (57).

Unlike for the static coordinate system, the expression for the intrinsic derivative in a moving coordinate system is order- and valence-dependent. For the covariant components of a vector the generalized intrinsic derivative is

$$\left.\frac{\delta w_i}{\delta t}\right|^{\chi} \equiv \left.\frac{\partial w_i}{\partial t}\right|_x + p^j \nabla_j w_i - \boxed{ w_j \nabla_i \mathcal{V}^j } , \tag{73}$$

[2]As a brief proof, we note

$$\left.\frac{\partial \boldsymbol{e}_i}{\partial t}\right|_x = \left.\frac{\partial}{\partial t}\right|_x \frac{\partial \boldsymbol{r}}{\partial x^i} = \frac{\partial}{\partial x^i} \left.\frac{\partial \boldsymbol{r}}{\partial t}\right|_x$$

just by reversing the order of the partial derivatives. Substituting from (64), this becomes

$$\left.\frac{\partial \boldsymbol{e}_i}{\partial t}\right|_x = \partial_i \left(\boldsymbol{\mathcal{V}} - \frac{d\boldsymbol{S}}{dt} \right) = \partial_i \boldsymbol{\mathcal{V}} = \boldsymbol{e}_j \nabla_i \mathcal{V}^j$$

using the fact that $\boldsymbol{S}$ is spatially constant together with the definition of the covariant derivative in (30).

[3]The first equality in (69) is true because the $\boldsymbol{e}_a$ are constant in time, and to verify the second equality we use (67) to write

$$\begin{aligned} \left.\frac{\partial \boldsymbol{\mathcal{V}}}{\partial t}\right|_y &= \left.\frac{\partial}{\partial t}\right|_x \left(\mathcal{V}^i \boldsymbol{e}_i \right) - \mathcal{V}^i \partial_i \boldsymbol{\mathcal{V}} \\ &= \left.\frac{\partial \mathcal{V}^i}{\partial t}\right|_x \boldsymbol{e}_i + \mathcal{V}^i \left(\left.\frac{\partial \boldsymbol{e}_i}{\partial t}\right|_x - \boldsymbol{e}_j \nabla_i \mathcal{V}^j \right) \end{aligned}$$

after using (30) in passing to the second equality; the term in parentheses then vanishes due to (68).

as derived in Appendix B. In comparison with (71) for the contravariant components, we see that the final term appears with the opposite sign and with a different contraction of indices. As shown in Appendix B, (71) and (73) are indeed the contravariant and covariant components of the same vector $d\boldsymbol{w}/dt|^{\chi}$. Despite this, neither the first nor third terms on their right-hand sides are individually the components of the same vector. The implications of this are discussed in section 6f.

For a second-order tensor $\boldsymbol{A}$ expressed in contravariant form, $\boldsymbol{A} = A^{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$, one finds the intrinsic derivative in a moving coordinate system to be

$$\left.\frac{\delta A^{ij}}{\delta t}\right|^{\chi} = \left.\frac{\partial A^{ij}}{\partial t}\right|_x + p^k \nabla_k A^{ij} + A^{kj}\nabla_k \mathcal{V}^i + A^{ik}\nabla_k \mathcal{V}^j. \tag{74}$$

Thus we see that when expressed in terms of the covariant derivative, the generalized intrinsic derivative exhibits a pattern that depends on the order and valence of the tensor components upon which it acts. This pattern is expressed most naturally in terms of the Lie derivative along $\mathcal{V}$, as discussed subseequently in section 6g.

As was the case for the covariant spatial derivative, the intrinsic derivative generalized to moving coordinate systems obeys the tensor transformation law

$$\frac{\delta w^i}{\delta t} = J^i_a \frac{\delta w^a}{\delta t}, \qquad \frac{\delta w_i}{\delta t} = J^a_i \frac{\delta w_a}{\delta t}$$
$$\frac{\delta w^a}{\delta t} = J^a_i \frac{\delta w^i}{\delta t}, \qquad \frac{\delta w_a}{\delta t} = J^i_a \frac{\delta w_i}{\delta t}. \tag{75}$$

Separately, it is compatible with the metric coefficients, and so indices can be raised and lowered across the derivative like so

$$\frac{\delta w^i}{\delta t} = g^{ij} \frac{\delta w_j}{\delta t}, \qquad \frac{\delta w_i}{\delta t} = g_{ij} \frac{\delta w^j}{\delta t}$$
$$\frac{\delta w^a}{\delta t} = g^{ab} \frac{\delta w_b}{\delta t}, \qquad \frac{\delta w_a}{\delta t} = g_{ab} \frac{\delta w^b}{\delta t}. \tag{76}$$

This result is standard for the derivative in fixed frames. The fact that it adds no complexity to time derivatives of zeroth-rank (scalar) tensors is shown in Appendix E, and the added complexity that comes with time derivatives of second rank tensors is shown in Appendix D. The result for moving settings is proven in section 6g.

g. The generalized material intrinsic derivative

The *generalized material intrinsic derivative* following fluid parcels for contravariant tensor components is then

$$\frac{\delta w^i}{\delta t} \equiv \left.\frac{\partial w^i}{\partial t}\right|_x + u^j \nabla_j w^i + \boxed{w^j \nabla_j \mathcal{V}^i}\,, \tag{77}$$

$$\frac{\delta w_i}{\delta t} \equiv \left.\frac{\partial w_i}{\partial t}\right|_x + u^j \nabla_j w_i - \boxed{w_j \nabla_i \mathcal{V}^j}\,, \tag{78}$$

found by choosing a family of curves χ_ξ as parcel trajectories relative to the coordinates such that $u^i = p^i|_\xi$ as in subsection 4c, but retaining the boxed terms due to the moving basis.

The first two terms on the right-hand side of (77) give the apparent rate of change of $\boldsymbol{w}$ following the flow $\boldsymbol{u}$ as observed from within the moving setting, measuring the contravariant components of $\boldsymbol{w}$. The final term tells us that at a fixed x location, the changing basis vectors interact with the vector field $\boldsymbol{w}$ in such a way as to act *as if* the setting velocity field $\mathcal{V}$ were being advected by $\boldsymbol{w}$. We term this effect *counter-advection*. Whereas in advection the gradient of a field is pushed past a fixed location by an advecting flow, in counter-advection the gradient of the setting velocity impinges upon a vector field at a fixed location.

An illuminating special case of counter-advection is that in which the vector field $\boldsymbol{w}$ is both spatially and temporally constant. In that case, temporal changes in the contravariant components w^i at fixed x locations must occur to counteract the fact that the basis vectors are also changing. The combined effect leaves the derivative of the $\boldsymbol{w}$ field unchanged.

h. A comment on the material derivative of tensors

As part of this work, we came across an ambiguity in standard terminology and notation that we felt worthy of mention. It is common to write the left-hand side of the momentum equation in a rigid frame (84) as $D\boldsymbol{u}/Dt$, where the D/Dt operator is defined to act on some field f as

$$\frac{Df}{Dt} \equiv \left(\frac{df}{dt}\right)_R = \left(\frac{\partial f}{\partial t}\right)_R + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\, f, \tag{79}$$

see for example Gill (1982, §4.1 & 4.5.1), Pedlosky (1987, §1.6), and Batchelor (2000, §3.2). This combination is often referred to as the material or total derivative. However, this terminology, and the notation D/Dt, are misleading because they do not indicate the crucial fact that the basis vectors of the moving setting are considered to be held fixed in evaluating the partial time derivative. We emphasize this idea as $(d/dt)_R$ instead.

Moreover, it is commonly explained that D/Dt gives the total rate of change of f that occurs as one moves through the fluid with the velocity $\boldsymbol{u}$. This is true in the case that f is a scalar field, or the coordinate system is in fact not moving. Yet when D/Dt is applied to a vector or tensor field in a moving setting, this interpretation is not correct. The material derivative of a vector $\boldsymbol{w}$ in a general moving coordinate system, presented in contravariant index notation form in (77), can be rewritten as

$$\frac{d\boldsymbol{w}}{dt} = \frac{\delta w^i}{\delta t}\boldsymbol{e}_i = \frac{D\boldsymbol{w}}{Dt} + w^i \left.\frac{\partial \boldsymbol{e}_i}{\partial t}\right|_x \tag{80}$$

by grouping the first two terms together. The total derivative following a fluid parcel trajectory d/dt and the "material derivative" D/Dt thus differ by a term dependent upon the time rate of change of the basis vectors. For example, in the rigid motion case, the material derivative of the relative velocity $\boldsymbol{u}$ is given by

$$\frac{d\boldsymbol{u}}{dt} = \frac{D\boldsymbol{u}}{Dt} + \boldsymbol{\Omega}\times\boldsymbol{u} \tag{81}$$

which differs from D/Dt by half of the Coriolis term.

The correct interpretation of D/Dt—the material derivative with the basis vectors held fixed—is that it represents the *apparent* material derivative of a vector or tensor field as documented by an observer in the moving coordinate system, for whom the temporal change of basis vectors will not be apparent. Another, equivalent definition of this operator would be the derivative having the same form as the material derivative of a scalar field. In the authors' opinion, the handling of this operator is a source of confusion that obscures the tensor nature of the quantities on which it acts, and makes the conversion between a fixed and moving setting more difficult.

5. The momentum equation in moving coordinates

The coordinate systems employed in geophysical fluid dynamics, such as isopycnal coordinates, are not often static in time. In this section we therefore develop the equations for conservation of momentum in arbitrary time-dependent coordinates using the generalized intrinsic material derivative from the previous section, and present interpretations of the novel terms which arise. These equations are directly relevant to ocean modeling.

a. Transformation to a rotating and translating frame

To motivate the development in this section we recall the standard presentation of the momentum equation in a rotating frame, as found in many textbooks, e.g. Vallis (2006, §2.1), Gill (1982, §4.5), Pedlosky (1987, §1.5–1.6), and Batchelor (2000, §3.2). See Morin (2008, §10) for a particularly thorough treatment. Although not normally done in GFD, for generality we allow the angular velocity of the moving setting to vary in time, and we also allow the moving setting to translate with a variable velocity. With these extensions, the moving setting exhausts the possibilities of rigid motion.

As before, let the y^a coordinates be embedded in an inertial frame, denoted I, that we take to be stationary. Now let the x^i coordinate system be temporarily limited to rigid motion so that it may be considered to be embedded in a rigidly rotating and translating frame of reference, denoted R. The absolute velocity $\boldsymbol{v}$ and velocity relative to the rotating frame $\boldsymbol{u}$ are then given by

$$\boldsymbol{v} = \frac{d\boldsymbol{s}}{dt}, \qquad \boldsymbol{u} = \left(\frac{d\boldsymbol{r}}{dt}\right)_R \tag{82}$$

where the meaning of the notation "$(\cdot)_R$" is that the basis vectors of the rotating frame R are to be held fixed as the derivative is being carried out. Note that the material derivative with the basis vectors held fixed is equivalent to the definition $\boldsymbol{u} \equiv dx^i/dt\big|_\xi\, \boldsymbol{e}_i$ employed previously in (48).

Conservation of momentum, expressed in terms of force per unit mass, appears in the inertial frame as

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\,\boldsymbol{v} = \boldsymbol{F} \tag{83}$$

where $\boldsymbol{F}$ is a generic forcing term, the details of which are not important here. In the rotating frame this becomes

$$\left(\frac{\partial \boldsymbol{u}}{\partial t}\right)_R + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u} = -\overbrace{2\boldsymbol{\Omega}\times\boldsymbol{u}}^{\text{Coriolis}} - \overbrace{\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})}^{\text{centrifugal}} - \overbrace{\frac{d\boldsymbol{\Omega}}{dt}}^{\text{Euler}}\times\boldsymbol{r} - \overbrace{\frac{d^2\boldsymbol{S}}{dt^2}}^{\text{translational}} + \boldsymbol{F}\,, \tag{84}$$

where the first four terms on the right-hand side are *apparent* or *fictitious forces* inferred within the rotating frame.

The first two apparent forces are well known in GFD. The Coriolis force appears to accelerate the fluid perpendicular to the direction of its apparent velocity $\boldsymbol{u}$ in the rotating frame, while the centrifugal force, which appears to push fluid parcels radially outwards, is generally subsumed into the gravitational portion of $\boldsymbol{F}$. The third and fourth apparent forces are typically neglected in geophysical fluid dynamics, but are included for completeness. The third apparent force, known as the Euler force (Marsden and Ratiu 1999, §8.6), the azimuthal force (Morin 2008, §10.2.4), or by no name (Batchelor 2000, §3.2), arises due to variability in the rotation rate Ω. We experience this in rotating amusement park rides, which apparently push us backwards as a carousel begins. It is important in planetary science problems such as tidally-locked moons. The final apparent force, the translational force, arises due to variability in the translational velocity $d\boldsymbol{S}/dt$ of the moving setting, and is the familiar "g-force" one feels inside an accelerating vehicle or elevator.

To transform the momentum equation from the inertial frame to the rotating frame, we first recognize that the absolute fluid velocity is the sum of the velocity observed in the rotating frame, the velocity induced by its rotation, and the translational velocity of the origin,

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{\Omega}\times\boldsymbol{r} + \frac{d\boldsymbol{S}}{dt}, \tag{85}$$

where the second term arises from the rate of change of the basis vectors induced by $\boldsymbol{\Omega}$, see e.g. Pedlosky (1987, §1.5) or Morin (2008, §9.5, §10.1). A second differentiation shows that accelerations in the inertial frame and the rigidly translating and rotating frame are related by

$$\frac{d\boldsymbol{v}}{dt} = \left(\frac{d\boldsymbol{u}}{dt}\right)_R + \boldsymbol{\Omega}\times\boldsymbol{u} + \boldsymbol{\Omega}\times\frac{d\boldsymbol{r}}{dt} + \frac{d\boldsymbol{\Omega}}{dt}\times\boldsymbol{r} + \frac{d^2\boldsymbol{S}}{dt^2}, \tag{86}$$

where again the second term arises from the rotation of the basis vectors. Noting that $\boldsymbol{r} = \boldsymbol{s} - \boldsymbol{S}$ and consequently $d\boldsymbol{r}/dt = \boldsymbol{v} - d\boldsymbol{S}/dt = \boldsymbol{u} + \boldsymbol{\Omega}\times\boldsymbol{r}$ from (85), this becomes

$$\frac{d\boldsymbol{v}}{dt} = \left(\frac{d\boldsymbol{u}}{dt}\right)_R + 2\boldsymbol{\Omega}\times\boldsymbol{u} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}) + \frac{d\boldsymbol{\Omega}}{dt}\times\boldsymbol{r} + \frac{d^2\boldsymbol{S}}{dt^2}. \tag{87}$$

Finally, the time derivative appearing on the right-hand side is expanded via the chain rule as

$$\left(\frac{d\boldsymbol{u}}{dt}\right)_R = \left(\frac{\partial\boldsymbol{u}}{\partial t}\right)_R + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u}. \tag{88}$$

Inserting these last two equations into (83) and moving terms to the right-hand-side yields (84).

b. The momentum equation in a moving setting

Having presented the momentum equation appropriate for a rigidly moving frame, which includes familiar terms arising from uniform rotation as well as some less familiar terms, we use a more rigorous derivation to find a version that remains appropriate when the setting deforms. Let the x coordinates again refer to a coordinate system in a deformable, moving setting. Applying the generalized material intrinsic derivative (77) to the components of the absolute velocity $\boldsymbol{v}$,

$$\frac{\delta v^i}{\delta t} = \left.\frac{\partial v^i}{\partial t}\right|_x + u^j\nabla_j v^i + v^j\nabla_j\mathcal{V}^i$$

Using $v^i = u^i + \mathcal{V}^i$,

$$\frac{\delta v^i}{\delta t} = \left.\frac{\partial u^i}{\partial t}\right|_x + u_j\nabla^j u^i + \boxed{2u_j\nabla^j\mathcal{V}^i + \mathcal{V}_j\nabla^j\mathcal{V}^i + \left.\frac{\partial\mathcal{V}^i}{\partial t}\right|_x} \tag{89}$$

The boxed terms are apparent forces that arise from a nonzero setting velocity $\boldsymbol{\mathcal{V}}$, and vanish otherwise.

The covariant form is different, because we use (73) instead of (77),

$$\frac{\delta v_i}{\delta t} = \left.\frac{\partial u_i}{\partial t}\right|_x + u_j\nabla^j u_i + \boxed{u^j\nabla_j\mathcal{V}_i - u^j\nabla_i\mathcal{V}_j - \mathcal{V}^j\nabla_i\mathcal{V}_j + \left.\frac{\partial\mathcal{V}^i}{\partial t}\right|_x} \tag{90}$$

In index notation, the momentum equation (per unit density) in the inertial frame is

$$\frac{\delta v^a}{\delta t} = \left.\frac{\partial v^a}{\partial t}\right|_y + v^b\nabla_b v^a = F^a, \tag{91}$$

where $\boldsymbol{F}$ is the force per unit density. From (89) this becomes in a moving, deforming setting

$$\left.\frac{\partial u^i}{\partial t}\right|_x + u^j\nabla_j u^i = \boxed{-2u^j\nabla_j\mathcal{V}^i - \mathcal{V}^j\nabla_j\mathcal{V}^i - \left.\frac{\partial\mathcal{V}^i}{\partial t}\right|_x} + F^i. \tag{92}$$

The boxed terms due to the setting velocity $\boldsymbol{\mathcal{V}}$ have been moved to the right-hand side and thus categorized as apparent forces, the meaning of we will explore shortly.

Given knowledge of the forces $\boldsymbol{F}$, this form of the momentum is numerically implementable for any moving coordinate system: material, isopycnal, pressure, or any other convenient choice. To accomplish this, it is only necessary to determine the form of the metric coefficients for that coordinate system, which will be presented explicitly in a companion paper (Feske et al. 2026). Thus (92) vastly expands the scope of possibilities over those afforded by the momentum equation in a rigid frame (84) which already has more apparent forces than included in most GFD textbooks. The companion paper presents concrete applications of (92) to various common coordinate systems employed in geophysical fluid mechanics. The remainder of this paper focuses on the interpretation of this equation.

Any second-order tensor can be expanded into two portions (Table 3): a symmetric part (which can be made *deviatoric* by further separating its isotropic part if desired), and an antisymmetric or "skew-symmetric" part. Applying this decomposition to a generic velocity gradient,

$$\nabla_j v^i = \overbrace{\frac{1}{2}\left(\nabla_j v^i + \nabla^i v_j\right)}^{\text{symmetric}\sim\text{strain,divergence}} + \overbrace{\frac{1}{2}\left(\nabla_j v^i - \nabla^i v_j\right)}^{\text{skew/spin}\sim\text{vorticity}} \tag{93}$$

These are the symmetric portion, which is associated with strain (the deviatoric part) and divergence (the isotropic part), and the antisymmetric portion, which is associated with spin and vorticity. See Table 3 for a comparison of symmetry operators between symbolic and index notation.

When both indices are lowered, we denote symmetries concisely as

$$\nabla_j v_i = \overbrace{\nabla_{(j} v_{i)}}^{\text{symmetric}} + \overbrace{\nabla_{[j} v_{i]}}^{\text{skew}}, \tag{94}$$

and similarly for raised indices—again, see Table 3. The skew-symmetric portion consists of the same information as the curl. One may write $\boldsymbol{e}^i \cdot \boldsymbol{\nabla} \times \boldsymbol{w} = \varepsilon^{ijk}\nabla_{[j} w_{k]}$, expressing the curl of any vector $\boldsymbol{w}$ in terms of the antisymmetric part of its gradient tensor.[4] The skew-symmetric velocity tensor $\nabla_{[j} v_{k]}$ corresponding to the vorticity, i.e. the curl of the velocity, is known as the *spin tensor* (Itskov 2015, p 61). The symmetric part $\nabla_{(j} v_{k)}$ is the *strain rate tensor*. We will refer to these operations on the setting velocity as the setting spin and setting strain rate.

With a small modification, involving adding and subtracting canceling terms, the momentum equation (92) achieves the alternate form

$$\begin{aligned}\left.\frac{\partial u^i}{\partial t}\right|_x + u^j \nabla_j u^i = &- \overbrace{2u_j \nabla^{[j}\mathcal{V}^{i]}}^{\text{generalized Coriolis}} - \overbrace{2u_j \nabla^{(j}\mathcal{V}^{i)}}^{\text{augmented Coriolis}} \\ &- \overbrace{\left(\mathcal{V}_j - \frac{\delta S_j}{\delta t}\right)\nabla^{[j}\mathcal{V}^{i]}}^{\text{generalized centrifugal}} - \overbrace{\left(\mathcal{V}_j - \frac{\delta S_j}{\delta t}\right)\nabla^{(j}\mathcal{V}^{i)}}^{\text{augmented centrifugal}} \\ &- \overbrace{\left.\frac{\partial \mathcal{V}^i}{\partial t}\right|_x - \frac{\delta S_j}{\delta t}\nabla^{[j}\mathcal{V}^{i]}}^{\text{generalized Euler-translational}} - \overbrace{\frac{\delta S_j}{\delta t}\nabla^{(j}\mathcal{V}^{i)}}^{\text{augmented translational}} \\ &+ F^i, \end{aligned} \tag{95}$$

in which the apparent forces become generalizations of the standard fictitious forces–Coriolis, centrifugal, and Euler and translational forces–all of which will be shown shortly to reduce to their familiar forms for the case of rigid motion. While this modification from (92) is not necessary (nor unique) for numerical implementation, it is useful in comparing the momentum equation in a moving setting to its rigid-frame counterpart. The *augmented* versions of these forces all vanish under the case of rigid motion because they depend on the setting strain rate, but can contribute important additional effects under a general setting velocity. As these augmented terms are unfamiliar in any case, we combine them into the *fantastical* force, as

$$\begin{aligned}\left.\frac{\partial u^i}{\partial t}\right|_x + u^j \nabla_j u^i = &- \overbrace{2u_j \nabla^{[j}\mathcal{V}^{i]}}^{\text{generalized Coriolis}} - \overbrace{\left(\mathcal{V}_j - \frac{\delta S_j}{\delta t}\right)\nabla^{[j}\mathcal{V}^{i]}}^{\text{generalized centrifugal}} \\ &- \overbrace{\left.\frac{\partial \mathcal{V}^i}{\partial t}\right|_x - \frac{\delta S_j}{\delta t}\nabla^{[j}\mathcal{V}^{i]}}^{\text{generalized Euler-translational}} - \overbrace{\left(2u_j + \mathcal{V}_j\right)\nabla^{(j}\mathcal{V}^{i)}}^{\text{fantastical}} + F^i. \end{aligned} \tag{96}$$

Furthermore, gathering the setting spin terms and strain rate terms separately is convenient as only the fantastical force, which depends on the setting strain rate, differs for contravariant and covariant momentum tensor representations (Table 5), which results from (71) and (73).

$$\begin{aligned}\left.\frac{\partial u_i}{\partial t}\right|_x + u^j \nabla_j u_i = &- \overbrace{2u^j \nabla_{[j}\mathcal{V}_{i]}}^{\text{generalized Coriolis}} - \overbrace{\left(\mathcal{V}^j - \frac{\delta S^j}{\delta t}\right)\nabla_{[j}\mathcal{V}_{i]}}^{\text{generalized centrifugal}} \\ &- \overbrace{\left.\frac{\partial \mathcal{V}_i}{\partial t}\right|_x - \frac{\delta S^j}{\delta t}\nabla_{[j}\mathcal{V}_{i]}}^{\text{generalized Euler-translational}} + \overbrace{\mathcal{V}^j \nabla_{(j}\mathcal{V}_{i)}}^{\text{fantastical}} + F_i. \end{aligned} \tag{97}$$

Table 5 presents (84) together with (96) and (97). The form of the fantastical force depends on whether the momentum is represented by a contravariant velocity as in (96) or a covariant velocity as in (97), because of (73) and (77). It is perhaps surprising that the different valences have such different governing equations. This will be discussed more deeply in section 6.

c. Recovery of the rigid frame equation

When the moving setting is limited to rigid rotation and translation, the setting velocity components are, by equations (85) and (51) we have

$$\mathcal{V}^i \to \mathcal{K}^i \equiv \varepsilon^{ijk}\Omega_j r_k + \frac{\delta S^i}{\delta t}, \tag{98}$$

from which it follows immediately that

$$\mathcal{K}^i - \frac{\delta S^i}{\delta t} = \varepsilon^{ijk}\Omega_j r_k = \boldsymbol{e}^i \cdot (\boldsymbol{\Omega} \times \boldsymbol{r}). \tag{99}$$

We choose the label $\mathcal{K}$ here because rigid coordinate motion is generated by *Killing vector fields*, as show in section 6g. From (96) we see that the derivatives we need to eval-

---

[4]This is readily found using the Hodge star operator $*(\cdot)$ of differential geometry, in terms of which one finds

$$\boldsymbol{\nabla} \times \mathcal{V} = (*d\mathcal{V}^\flat)^\sharp = \boldsymbol{e}^i \frac{1}{2}\varepsilon_i{}^{jk}\left(\nabla_j \mathcal{V}_k - \nabla_k \mathcal{V}_j\right) = \boldsymbol{e}_i \varepsilon^{ijk}\nabla_{[j}\mathcal{V}_{k]}.$$

where $d\mathcal{V}^\flat$ is the exterior derivative of the 1-form associated with $\mathcal{V}$ by the metric tensor. See, e.g., §4.2 of Marsden and Ratiu (1999), §2.9 of Carroll (2004), §36.5 of Needham (2021), or §3c of Lilly et al. (2024) for further details.

TABLE 5. Conservation of momentum in the absence of external forces written in symbolic form in a rigid frame rotating with potentially variable angular velocity $\boldsymbol{\Omega}$ and translating with velocity $d\boldsymbol{S}/dt$, on the upper line, and written in index notation form for a general moving, deforming setting, on the lower line, where u^i is a relative velocity in the moving, deforming coordinates x^i. The lower line is far more general because the moving coordinate system is not constrained to be rigid, but rather moves *and* deforms according to the setting velocity field $\boldsymbol{\mathcal{V}} = \mathcal{V}^i \boldsymbol{e}_i$. The three terms in boxes are apparent forces that generalize those encountered in the rigid case. When the setting velocity $\boldsymbol{\mathcal{V}} = \boldsymbol{V}$ is chosen to arise from rigid translation and rotation, such that $\boldsymbol{V} = \boldsymbol{\Omega} \times \boldsymbol{r} + d\boldsymbol{S}/dt$, the lower line reduces to the upper line. Additional forces can be applied to the right side as well.

Equation	Tendency	Advection	Coriolis	Centrifugal	Euler & Translational	Fantastical	Forces
(84)	$\left(\frac{\partial \boldsymbol{u}}{\partial t}\right)_R$	$+(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u} =$	$-2\boldsymbol{\Omega}\times\boldsymbol{u}$	$-\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})$	$-\frac{d\boldsymbol{\Omega}}{dt}\times\boldsymbol{r}-\frac{d^2\boldsymbol{S}}{dt^2}$	$+\boldsymbol{0}$	$\boldsymbol{F}$
(96)	$\left.\frac{\partial u^i}{\partial t}\right\rvert_x$	$+u^j\nabla_j u^i =$	$\boxed{-2u_j\nabla^{[j}\mathcal{V}^{i]}}$	$\boxed{-\left(\mathcal{V}_j-\frac{\delta S_j}{\delta t}\right)\nabla^{[j}\mathcal{V}^{i]}}$	$\boxed{-\left.\frac{\partial \mathcal{V}^i}{\partial t}\right\rvert_x-\frac{\delta S_j}{\delta t}\nabla^{[j}\mathcal{V}^{i]}}$	$\boxed{-\left(2u_j+\mathcal{V}_j\right)\nabla^{(j}\mathcal{V}^{i)}}$	F^i
(97)	$\left.\frac{\partial u_i}{\partial t}\right\rvert_x$	$+u^j\nabla_j u_i =$	$\boxed{-2u^j\nabla_{[j}\mathcal{V}_{i]}}$	$\boxed{-\left(\mathcal{V}^j-\frac{\delta S^j}{\delta t}\right)\nabla_{[j}\mathcal{V}_{i]}}$	$\boxed{-\left.\frac{\partial \mathcal{V}_i}{\partial t}\right\rvert_x-\frac{\delta S^j}{\delta t}\nabla_{[j}\mathcal{V}_{i]}}$	$\boxed{+\mathcal{V}^j\nabla_{(j}\mathcal{V}_{i)}}$	F_i

uate are under rigid motions just

$$
\begin{aligned}
&\nabla^j\mathcal{K}^i = \nabla^{[j}\mathcal{K}^{i]} = \varepsilon^{i\ell k}\nabla^j\left(\Omega_\ell r_k\right) = \varepsilon^{i\ell k}\delta^j_k\Omega_\ell = \varepsilon^{i\ell j}\Omega_\ell,\\
&\nabla^{(j}\mathcal{K}^{i)} = 0\,,\\
&\left.\frac{\partial\mathcal{K}^i}{\partial t}\right\rvert_x = \left.\frac{\partial^2 S^i}{\partial t^2}\right\rvert_x + \varepsilon^{ijk}\left.\frac{\partial}{\partial t}\right\rvert_x \left(\Omega_j s_k\right).
\end{aligned}
\tag{100}
$$

The moving frame is rigid, so the spatial derivatives of S^i and Ω_j are zero. The partial time derivative of r_k when holding x fixed is simply zero; all of the change in $\partial\boldsymbol{r}/\partial t|_x$ is encapsulated in the change of the basis vectors. Metrinilic and other properties from section 3b and Table A1 are also used. Shifting of indices from upper to lower position can occur later as the situation demands by the index juggling operations. Importantly, the velocity gradient tensor is antisymmetric in this case. This will mean that the fantastical force is zero for rigid rotation and translation in (96) and (97).

Thus, we find the moving rigid terms (Table 5, upper row) as

$$
\begin{aligned}
-2u_j\nabla^{[j}\mathcal{K}^{i]} &= -2u_j\varepsilon^{i\ell j}\Omega_\ell = \boldsymbol{e}^i\cdot(-2\boldsymbol{\Omega}\times\boldsymbol{u}),\\
-\left(\mathcal{K}_j-\frac{\delta S_j}{\delta t}\right)\nabla^{[j}\mathcal{K}^{i]} &= -\left(\varepsilon_{jk\ell}\Omega^k r^\ell\right)\varepsilon^{imj}\Omega_m\\
&= \left(r^i\Omega^k - r^k\Omega^i\right)\Omega_k\\
&= \boldsymbol{e}^i\cdot(-\boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\boldsymbol{r}),\\
-\left(2u_j+\mathcal{K}_j\right)\nabla^{(j}\mathcal{K}^{i)} &= 0,\\
\mathcal{K}_j\nabla^{(j}\mathcal{K}^{i)} &= 0,\\
-\left.\frac{\partial\mathcal{K}^i}{\partial t}\right\rvert_x - \frac{\delta S_j}{\delta t}\nabla^{[j}\mathcal{K}^{i]} &= -\varepsilon^{ijk}\frac{\delta\Omega_j}{\delta t}r_k - \frac{\delta^2 S^i}{\delta t^2}\\
&= \boldsymbol{e}^i\cdot\left(-\frac{d\boldsymbol{\Omega}}{dt}\times\boldsymbol{r}-\frac{d^2\boldsymbol{S}}{dt^2}\right).
\end{aligned}
\tag{101}
$$

Of course, under more complicated setting velocities $\boldsymbol{\mathcal{V}}$ instead of $\boldsymbol{\mathcal{K}}$ from (98), these simplifications do not occur.

The derivations in the previous section, summarized in Table 5, show that both the Coriolis force and centrifugal force are closely related to advection terms, the former arising from the spin part of the advection of the setting velocity $\boldsymbol{\mathcal{V}}$ by the relative velocity $\boldsymbol{u}$ plus the spin part of the boxed term in (77), and the latter resembles self-advection of the spin portion of the setting velocity gradient, $\boldsymbol{\Omega}\times\boldsymbol{r}$. However, these fictitious forces arise mostly because of the boxed terms in (77) and (78). These terms are the key distinction between the form of the inertial frame material derivative, which is normally considered "advection" and is discussed in section 4c and the correct form for the generalized intrinsic material derivative of the momentum $\delta v^i/\delta t$.

The fantastical force term arises—in both the contravariant and covariant momentum equations—only when the setting velocity gradient tensor has a non-zero symmetric part, which doesn't occur under rigid rotation and translation. Finally, note the brevity index notation allows: this section is a fraction of a page long while our equivalent symbolic notation version of these proofs took over two pages.

6. Discussion

In this section we discuss the generalized apparent forces and the force balances in the context of the literature, especially the works of Oldroyd, Salmon, and Holm and their collaborators, with particular attention to comparing the generalized intrinsic derivative to related operators.

a. The apparent acceleration

We can think of observing velocity as the process of marking the coordinates of successive positions over a small time interval and taking their difference. In two dimensions the coordinates can be visualized as lines drawn on a deformable rubber sheet. If the coordinate lines are moving, this is unknown to an observer who has only these lines themselves as a reference. Similarly, acceleration is observed by a second difference of these position marks, and again, contributions to the acceleration that arise from the motion of the coordinates are invisible to an observer in the moving setting. Separately, acceleration may be measured through an accelerometer, or felt viscerally in one's body. When in a moving setting, the measured, or felt accelerations that would not occur in a Newtonian rigid frame constitute the apparent forces, precisely the boxed terms in a moving setting (92).

The two terms on the left-hand side of (92) describe the acceleration observed from the moving setting as one moves with the relative velocity $\boldsymbol{u}$, and records contravariant components of velocity. The first, $\partial u^i/\partial t\big|_x$, is local rate of change of the contravariant components of velocity relative to the moving setting observed at a fixed coordinate location in that setting. The second, $u^j\nabla_j u^i$, is ith component of the advective term $\boldsymbol{e}^i\cdot(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u}$, the rate of change of relative velocity $\boldsymbol{u}$ that arises from moving through gradients of the relative velocity at the relative velocity itself. Contributions to the absolute velocity $\boldsymbol{v}$ that involve the setting velocity $\boldsymbol{\mathcal{V}}$ will not be apparent to an observer who quantifies motion based only on references to the setting coordinates. Consequently, felt accelerations that depend upon $\boldsymbol{\mathcal{V}}$ will seem to the observer to be unrelated to motion; the observer will instead conclude that these terms must be forces: the fictitious and fantastical forces.

An important point is that unlike for a rigid frame, in a deforming setting the partitioning of the material acceleration $d\boldsymbol{v}/dt$ into an apparent acceleration and apparent forces is dependent upon whether an observer chooses to record velocity and acceleration through through contravariant or covariant components. This distinction, which appears only in the fantastical force, stems from the generalized intrinsic material derivative—compare (77) versus (78). This derivative was constructed precisely to preserve the dynamics of the velocity tensor in both representations, so it usually suffices to consider only one representation (we typically choose contravariant). The distinction between the contravariant and covariant representations and why they differ in the generalized intrinsic material derivative will be discussed in a geometric context in section 6f.

b. The generalized Coriolis force

Table 5 shows that the Coriolis force is a disguised advection term arising from the relative flow $\boldsymbol{u}$ advecting the particular setting velocity $\boldsymbol{\mathcal{K}} = \boldsymbol{\Omega}\times\boldsymbol{r} + d\boldsymbol{S}/dt$. To explore the Coriolis generalization, we first mention several facts about the setting velocity of a rigidly rotating frame (98), namely

$$\mathcal{K}^i = \varepsilon^{ijk}\Omega_j r_k + \frac{\delta S^i}{\delta t}, \tag{102}$$

$$\nabla^j\mathcal{K}^i = \nabla^{[j}\mathcal{K}^{i]} = \varepsilon^{i\ell j}\Omega_\ell, \tag{103}$$

$$\varepsilon_{ijk}\nabla^j\mathcal{K}^k = \varepsilon_{ijk}\varepsilon^{k\ell j}\Omega_\ell = 2\Omega_i\ , \tag{104}$$

$$\nabla^{(j}\mathcal{K}^{i)} = 0, \tag{105}$$

$$\nabla^i\mathcal{K}_i = 0. \tag{106}$$

meaning that this velocity field is nondivergent (106) and has a spatially uniform vorticity (104). Thus the velocity gradient tensor is just the spin tensor, and the augmented apparent force is zero. Note that (104) is a just an application of (19) that results in the coefficient of 2. Because $d\boldsymbol{S}/dt$ is a spatially constant vector field, it does not contribute to any of the spatial derivatives.

The above results suggest that within a general setting, a local analogue to the standard Coriolis force can be found through an expansion of the gradient tensor of the setting velocity, $\nabla^j\mathcal{V}^i$. With this decomposition, the generalized Coriolis force then likewise consists of two parts formed from the decomposed setting velocity gradient

$$-2u^j\nabla_j\mathcal{V}^i = -2u_j\overbrace{\nabla^{[i}\mathcal{V}^{j]}}^{\text{skew/spin}} - 2u_j\overbrace{\nabla^{(i}\mathcal{V}^{j)}}^{\text{symmetric/strain}} \tag{107}$$

The skew-proportional term will be called the *generalized Coriolis force*. This is a Coriolis-like force that is different at every point in space, as it arises from the local spin associated with the setting velocity $\boldsymbol{\mathcal{V}}$. This term, like the standard Coriolis force, is always perpendicular to the advecting relative velocity $\boldsymbol{u}$, as $u^i u^j \nabla_{[i}\mathcal{V}_{j]} = 0$ by the symmetry of $u^i u^j$ and skew symmetry of the setting spin.[5] This implies that this term has no acceleration in the direction of $\boldsymbol{u}$ and does no work in the relative velocity's kinetic energy budget. It is proportional to the relative velocity, and thus has no effect when the fluid is fixed in relation to the coordinates. These properties resemble the ordinary Coriolis effect.

The term that depends on the symmetric portion of the velocity gradient on the right-hand side of (107) is quite different, and it is our first fantastical force. It will be examined in section 6e below.

A useful calculation for any vector field w_j is the divergence of its contraction with the setting spin and strain rate tensor,

$$\nabla_i\left(w_j\nabla^{[j}\mathcal{V}^{i]}\right) = \left(\nabla_i w_j\right)\left(\nabla^{[j}\mathcal{V}^{i]}\right) + w_j\nabla_i\left(\nabla^{[j}\mathcal{V}^{i]}\right), \tag{108}$$

$$\nabla_i\left(w_j\nabla^{(j}\mathcal{V}^{i)}\right) = \left(\nabla_i w_j\right)\left(\nabla^{(j}\mathcal{V}^{i)}\right) + w_j\nabla_i\left(\nabla^{(j}\mathcal{V}^{i)}\right). \tag{109}$$

The curl of these fields simplifies a bit more than the divergence because one of the two terms in the symmetrization decomposition vanishes, and the remaining one features the spin of the setting velocity.

$$\varepsilon_{mni}\nabla^n\left(w_j\nabla^{[j}\mathcal{V}^{i]}\right) = \varepsilon_{mni}\left(\nabla^n w_j\right)\left(\nabla^{[j}\mathcal{V}^{i]}\right) + w_j\nabla^j\varepsilon_{mni}\nabla^{[n}\mathcal{V}^{i]}, \tag{110}$$

$$\varepsilon_{mni}\nabla^n\left(w_j\nabla^{(j}\mathcal{V}^{i)}\right) = \varepsilon_{mni}\left(\nabla^n w_j\right)\left(\nabla^{(j}\mathcal{V}^{i)}\right) + \frac{1}{2}w_j\nabla^j\left(\varepsilon_{mni}\nabla^{[n}\mathcal{V}^{i]}\right). \tag{111}$$

When $w_j = -2u_j$ is used, (108) and (110) become the divergence and curl of the generalized Coriolis force,

$$\nabla_i\left(-2u_j\nabla^{[j}\mathcal{V}^{i]}\right) = -2\left(\nabla_i u_j\right)\left(\nabla^{[j}\mathcal{V}^{i]}\right) - 2u_j\nabla_i\left(\nabla^{[j}\mathcal{V}^{i]}\right), \tag{112}$$

and

$$\varepsilon_{mni}\nabla^n\left(-2u_j\nabla^{[j}\mathcal{V}^{i]}\right) = -2\varepsilon_{mni}\left(\nabla^n u_j\right)\left(\nabla^{[j}\mathcal{V}^{i]}\right) - 2u_j\nabla^j\varepsilon_{mni}\nabla^{[n}\mathcal{V}^{i]}. \tag{113}$$

[5] Any symmetric tensor doubly-contracted with any antisymmetric tensor is zero.

c. The generalized centrifugal force

The centrifugal force is also of the form of an advection term in disguise (Table 5). As with the Coriolis force, we can isolate a generalized centrifugal force that shares most properties with the standard centrifugal force, with the key ones being: it is proportional to the antisymmetric part of the velocity gradient tensor and it is conservative, i.e., it can be represented by the gradient of a fixed scalar field and relatedly it has no curl.

The standard rigid-frame centrifugal force has a divergence that is nonnegative and constant in space,

$$\begin{aligned}-\boldsymbol{\nabla}\cdot(\boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\boldsymbol{r}) &= \nabla_i\left(r^i\Omega^k - r^k\Omega^i\right)\Omega_k\\ &= \Omega_k\Omega^k\delta^i_i - \Omega_k\Omega^i\delta^k_i\\ &= \Omega_k\Omega^k(3-1)\\ &= 2\Omega^k\Omega_k.\end{aligned} \tag{114}$$

Its curl is zero,

$$\begin{aligned}-\boldsymbol{e}^i\cdot\boldsymbol{\nabla}\times(\boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\boldsymbol{r}) &= \varepsilon^{i\ell m}\nabla_\ell\left[\left(r_m\Omega^k - r^k\Omega_m\right)\Omega_k\right]\\ &= \varepsilon^{i\ell}{}_{\ell}\Omega^k\Omega_k - \varepsilon^{i\ell m}\Omega_m\Omega_\ell\\ &= 0.\end{aligned} \tag{115}$$

These properties of the standard, rigid-frame centrifugal force imply that it is a conservative force and can be written as the gradient of a potential—a crucial step in modeling and theory, because it allows the combination of the centrifugal force and gravity into the geopotential. To find the potential corresponding to the rigid centrifugal force, we begin with (102) and (103), and use these in the form of the Coriolis force from Table 5:

$$\begin{aligned}-\left(\mathcal{K}_j - \frac{\delta S_j}{\delta t}\right)\nabla^{[i}\mathcal{K}^{j]} &= -\left(\varepsilon_{jmk}\Omega^m r^k\right)\varepsilon^{i\ell j}\Omega_\ell\\ &= -\left(\varepsilon_{jmk}\Omega^m r^k_\perp\right)\varepsilon^{i\ell j}\Omega_\ell\\ &= \Omega^\ell\Omega_\ell r^i_\perp - \Omega^i \cancelto{0}{r^\ell_\perp\Omega_\ell}\\ &= \frac{1}{2}\nabla^i\left(\Omega^\ell\Omega_\ell r^j_\perp r^\perp_j\right)\\ &= \nabla^i\left(\frac{1}{2}\Omega^2 r^2_\perp\right).\end{aligned} \tag{116}$$

Where $\boldsymbol{r}_\perp$ is defined so that $\boldsymbol{\Omega}\times\boldsymbol{r} = \boldsymbol{\Omega}\times\boldsymbol{r}_\perp$ and $\boldsymbol{\Omega}\cdot\boldsymbol{r}_\perp = 0$, and we use the identity (19) and $\nabla^i r_j = \delta^i_j$.

The generalized centrifugal force shares also has a nonnegative term in its divergence, but it is not spatially uniform. When $w_j = \left(\frac{\delta S_j}{\delta t} - \mathcal{V}_j\right)$ is used in (108) and (110), the divergence and curl of the generalized and augmented centrifugal forces can be found. The divergence is

$$\nabla_i\left(\left(\frac{\delta S_j}{\delta t}-\mathcal{V}_j\right)\nabla^{[j}\mathcal{V}^{i]}\right)=\left(\nabla_{[i}\mathcal{V}_{j]}\right)\left(\nabla^{[i}\mathcal{V}^{j]}\right)+\left(\frac{\delta S_j}{\delta t}-\mathcal{V}_j\right)\nabla_i\left(\nabla^{[j}\mathcal{V}^{i]}\right). \tag{117}$$

The divergence begins with a doubly-contracted nonnegative contribution, similar to the rigid frame case. The other contribution consists of second derivatives of the setting velocity, so this term is focused where the setting velocity is anomalous from its surroundings. Examples where these terms might be important are strong fronts in isopycnal coordinates, steep topography in terrain-following coordinates, etc. In the rigid rotation case, this effect is absent as the spin of the setting velocity is constant in space by (103) and its strain and divergence are zero by (105).

The curl of the centrifugal force is

$$\begin{aligned}&\varepsilon_{mni}\nabla^n\left[\left(\frac{\delta S_j}{\delta t}-\mathcal{V}_j\right)\nabla^{[j}\mathcal{V}^{i]}\right]\\&=-\varepsilon_{mni}\left(\nabla^n\mathcal{V}_j\right)\left(\nabla^{[j}\mathcal{V}^{i]}\right)+\left(\frac{\delta S_j}{\delta t}-\mathcal{V}_j\right)\varepsilon_{mni}\nabla^n\nabla^{[j}\mathcal{V}^{i]},\end{aligned} \tag{118}$$

which may be nonzero for any setting with nonuniform setting velocity or spin, so the centrifugal force is not guaranteed to be conservative as it is in the rigid frame case. A Helmholtz decomposition can be used to isolate the conservative portion of this centrifugal force in (118), but it is not simply related to either the spin or strain rate portions of the force, as the final term in (118) indicates that a gradient in the spin provides a curl in the force, i.e., a non-conservative force.

d. Euler and translational force

These forces are not typically included in GFD, because translational accelerations are neglected and the rotation rate of the rigid frame is normally taken to be constant. However if they are nonzero, whether in rigid frames or deforming ones they are necessary. The augmented translational contribution cancels part of the augmented contribution from the centrifugal force in all cases, simplifying the fantastical force to be independent of $\boldsymbol{S}$.

e. The fantastical force

The fantastical force is unfamiliar to those accustomed to rotations and translations of rigid frames. We found contributions to this force from the terms that contribute to the Coriolis and centrifugal forces. As it is proportional to the symmetric part of the velocity gradient, it is only present when the setting velocity has divergence and/or strain.

$$-\left(2u_j+\mathcal{V}_j\right)\nabla^{(j}\mathcal{V}^{i)} \tag{119}$$

The divergence of the two contributions to the contravariant fantastical force are found from (109) to be

$$\nabla_i\left(-2u_j\nabla^{(j}\mathcal{V}^{i)}\right)=-2\left(\nabla_{(i}u_{j)}\right)\nabla^{(j}\mathcal{V}^{i)}-2u_j\nabla_i\left(\nabla^{(j}\mathcal{V}^{i)}\right), \tag{120}$$

$$\nabla_i\left(-\mathcal{V}_j\nabla^{(j}\mathcal{V}^{i)}\right)=-\nabla_{(i}\mathcal{V}_{j)}\nabla^{(i}\mathcal{V}^{j)}-\mathcal{V}_{(j}\nabla_{i)}\left(\nabla^{(j}\mathcal{V}^{i)}\right). \tag{121}$$

The first term on the right hand side of (120) is unaffected by spin in the relative velocity. A nonpositive definite contribution is the first term on the right side of (121), and the other three contributions are focused where the setting velocity is anomalous from its surroundings.

The curl of the two contributions to the fantastical force are found from (111) with $w_j=\left(-2u_j-\mathcal{V}_j\right)$ as

$$\begin{aligned}\varepsilon_{mni}\nabla^n\left(\left(-2u_j-\mathcal{V}_j\right)\nabla^{(j}\mathcal{V}^{i)}\right)=&\\-2\varepsilon_{mni}\left(\nabla^n u_j\right)\left(\nabla^{(j}\mathcal{V}^{i)}\right)&\\-\frac{\varepsilon_{mni}}{2}\left(\nabla^n\mathcal{V}_j\right)\left(\nabla^j\mathcal{V}^i\right)&\\-\left(u_j+\frac{\mathcal{V}_j}{2}\right)\nabla^j\varepsilon_{mni}\nabla^n\mathcal{V}^i.&\end{aligned} \tag{122}$$

These results[6] suggest there is little reason to expect the fantastical force to be zero or be conservative for most interesting deforming coordinate systems, except when $\nabla^{(j}\mathcal{V}^{i)}=0$. The terms proportional to u vanish for the covariant fantastical force, but the other terms remain (with the opposite sign) – see equations (96) and (97). Many other papers have used essentially the same form of the Coriolis and centrifugal forces in deforming coordinates as in rigid ones when other assumptions (e.g., hydrostasy, strong stratification) are employed (e.g., Bleck 2002); presumably, those forces are similar when the coordinate surfaces are nearly aligned with geopotential surfaces. This paper does not examine the size of the different contributions; that will come in a later asymptotic analysis.

f. Contravariant versus covariant representations

Our discussion of the fantastical forces has repeatedly mentioned that the momentum equation in a moving setting—but not the resulting dynamics—depends upon the choice of a contravariant or covariant representation of momentum. To see this another way, we borrow a concept

[6] The second term on the right is only one-half of the symmetrized setting velocity gradient, because the other half vanishes as with its preceding factor, it is symmetric in $n \Leftrightarrow i$ while the Levi-Civita tensor is antisymmetric.

from continuum mechanics: the *upper* and *lower convected time derivatives*, written respectively as

$$\overset{\triangledown}{\boldsymbol{w}} = \frac{d\boldsymbol{w}}{dt} - \boldsymbol{w}\,(\boldsymbol{\nabla}\otimes\boldsymbol{v}) \tag{123}$$

$$\overset{\vartriangle}{\boldsymbol{w}} = \frac{d\boldsymbol{w}}{dt} + \boldsymbol{w}\,(\boldsymbol{\nabla}\otimes\boldsymbol{v})^T \tag{124}$$

These derive ultimately from Oldroyd (1950), and give the apparent rates of change of the vector $\boldsymbol{w}$ observed when moving at the fluid velocity $\boldsymbol{v}$. The triangles "▽" and "△" indicate the valence of the vector components to which the partial time derivative is applied, the former applies when measuring w^i or w^a, and the latter when measuring w_i or w_a. This notation is common in continuum mechanics (e.g., Dimitrienko 2011; Hinch and Harlen 2021). See Appendix F for an explicit, component-wise definition of these derivatives applicable to arbitrary tensor fields. Rearranging, we can write

$$\frac{d\boldsymbol{w}}{dt} = \overset{\triangledown}{\boldsymbol{w}} + \boldsymbol{w}\,(\boldsymbol{\nabla}\otimes\boldsymbol{v}) \tag{125}$$

$$\frac{d\boldsymbol{w}}{dt} = \overset{\vartriangle}{\boldsymbol{w}} - \boldsymbol{w}\,(\boldsymbol{\nabla}\otimes\boldsymbol{v})^T \tag{126}$$

for the contravariant (77) and covariant (78) versions of the material derivative as expressed in symbolic form, respectively. Note that because these derivatives deal particularly with convected coordinates (which we would call *advected* or Lagrangian coordinates), we must make the particular choice $\boldsymbol{\mathcal{V}} = \boldsymbol{v}$.

When these operators are applied to the local velocity $\boldsymbol{u}$, this presentation shows clearly that, depending on which set of components we measure, there are two possible partitionings of the material acceleration $d\boldsymbol{u}/dt$ into an apparent local rate of change—the first terms on the right hand side—and apparent forces, the negative of the second term. This choice is arbitrary, and does not affect the actual dynamics. The apparent forces are the same if and only if the transformation is rigid, that is, the gradient tensor of the setting velocity $\boldsymbol{\nabla}\otimes\boldsymbol{v}$ is everywhere antisymmetric.[7] In this rigid case, the apparent local accelerations will also be the same, as shown in Appendix C. This is why we distinguish the fictitious forces (which are constructed from only the setting spin part) from the fantastical forces (which are constructed from the setting strain rate and divergence) as shown in Table 5.

The apparent forces arising in rigidly moving frames, being independent of the valence of the tensor's representation, are of a different nature than those which arise more generally in the presence of deformation. To understand this, consider that a moving observer will feel their total acceleration $d\boldsymbol{v}/dt$, and will not be able to viscerally distinguish between its various components. An observer standing in a rotating room, however, feels an outward acceleration, takes note of the fact that they are not in motion relative to the room, and infers the presence of a force acting outwards. In this way apparent forces are inferred from felt accelerations and measured motion together. The above result implies that when the setting is not rigid, the inference of apparent forces from felt accelerations no longer has a unique result

[7] $\boldsymbol{\nabla}\otimes\boldsymbol{v} = -\left(\boldsymbol{\nabla}\otimes\boldsymbol{v}\right)^T$, or equivalently $\nabla^i v^j = \nabla^{[i} v^{j]}$ and $\nabla^{(i} v^{j)} = 0$.

g. Generalized intrinsic derivatives in terms of Lie derivatives

A generalized advection operator can be rewritten in terms of a Lie derivative (e.g., Holm 2015, 2025) acting along $\boldsymbol{\mathcal{V}}$. Some properties of the Lie derivative and an example of its use in taking the derivative of a Reynolds stress are in Appendix D. In our case, this means that the generalized intrinsic material derivative (71) is equivalent to a Lie derivative form:

$$\left.\frac{\delta}{\delta t}\right|^{\chi} = \left(\left.\frac{\partial}{\partial t}\right|_x + q^j \nabla_j - \mathcal{L}_{\mathcal{V}} \right). \tag{127}$$

a form which is now independent of the valence and order of the tensor components of its argument. For the contravariant components of a vector, covariant components of vector, and contravariant components of a second-order tensor, the Lie derivative is given respectively by

$$\mathcal{L}_{\mathcal{V}} w^i = \mathcal{V}^j \partial_j w^i - w^j \partial_j \mathcal{V}^i, \tag{128}$$

$$\mathcal{L}_{\mathcal{V}} w_i = \mathcal{V}^j \partial_j w_i + w_j \partial_i \mathcal{V}^j, \tag{129}$$

$$\mathcal{L}_{\mathcal{V}} A^{ij} = \mathcal{V}^k \partial_k A^{ij} - A^{kj} \partial_k \mathcal{V}^i - A^{ik} \partial_k \mathcal{V}^j, \tag{130}$$

a pattern that accounts for the valence dependence of the generalized intrinsic derivative in equations (71), (73), and (74). The general form for the Lie derivative may be found in (B.18) of Carroll (2004), or likely in any text on differential geometry or general relativity.

Due to the symmetry of the Christoffel symbols, the partial derivatives may equivalently be written as covariant derivatives, since the all components dependent on the Christoffel symbols will cancel. We then note that the relationship $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{\mathcal{V}}$ given in (51), a consequence of the chain rule, is a special case of the more general relationship

$$\boldsymbol{q} = \boldsymbol{p} + \boldsymbol{\mathcal{V}} \tag{131}$$

between the parametric velocities $\boldsymbol{p}$ and $\boldsymbol{q}$ along a curve χ as defined in (53). With $q^j = p^j + \mathcal{V}^j$ and together with (128)–(130), the expressions for the generalized intrinsic derivative given previously in (71), (73), and (74) are all recovered from the Lie derivative form (127). Similarly

$$\frac{\delta}{\delta t} = \left(\left.\frac{\partial}{\partial t}\right|_x + v^j \nabla_j - \mathcal{L}_{\mathcal{V}} \right) \tag{132}$$

is the Lie derivative form of the material intrinsic derivative, obtained from (74) by setting $\boldsymbol{q} = \boldsymbol{v}$. We can see now that in the moving setting, instead of simple advection $(\boldsymbol{u} \cdot \boldsymbol{\nabla})$ by the observed velocity $\boldsymbol{u}$, each part of $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{\mathcal{V}}$ effects a different sort of advection: standard, covariant advection by the Newtonian frame velocity $\boldsymbol{v}$ and Lie advection opposite the setting velocity $\boldsymbol{\mathcal{V}}$, which is the source of the apparent forces.

The appearance of the Lie derivative in the generalized intrinsic derivative can be understood as follows: the criterion for a well-behaved moving coordinate system x^i is that the setting velocity $\boldsymbol{\mathcal{V}}$ be a vector field that generates a diffeomorphism—that is, an invertible, one-to-one mapping—from space onto itself, as illustrated in Figure 2. By definition, the Lie derivative $\mathcal{L}_{\mathcal{V}}$ is the derivative that measures how a tensor field changes under the diffeomorphism generated by $\boldsymbol{\mathcal{V}}$.

The Lie derivative formulation can be used to verify the metric compatibility of the generalized intrinsic derivative. The combination

$$\left.\frac{\partial}{\partial t}\right|_x - \mathcal{L}_{\mathcal{V}} \tag{133}$$

measures the difference between the observed change with time at a fixed x coordinate location and the change due only to the rearrangement of coordinates induced by $\boldsymbol{\mathcal{V}}$. This operation is metric compatible because the actions of $\partial/\partial t|_x$ and $\mathcal{L}_{\mathcal{V}}$ on the metric coefficients lead to the same value,

$$\left.\frac{\partial g_{ij}}{\partial t}\right|_x = \mathcal{L}_{\mathcal{V}} g_{ij} = 2\nabla_{(i}\mathcal{V}_{j)} \tag{134}$$

and hence (133) applied to g_{ij} vanishes. Together with the metric compatibility of the covariant derivative this implies metric compatibility of the generalized intrinsic derivative from (127),

$$\left.\frac{\delta g_{ij}}{\delta t}\right|^{\mathcal{X}} = 0. \tag{135}$$

This property is important because it allows us to freely raise and lower indices:

$$\left.\frac{\delta w_i}{\delta t}\right|^{\mathcal{X}} = g_{ij} \left.\frac{\delta w^j}{\delta t}\right|^{\mathcal{X}}. \tag{136}$$

Metric compatibility of $\delta/\delta t|^{\mathcal{X}}$ is expected since this operator was constructed to be tensorial, but it is nevertheless reassuring to demonstrate directly. The form of the intrinsic derivative for covariant components is derived without Lie derivatives in Appendix B.

As an aside, we mention that (134) also means that if $\nabla_{(i}\mathcal{V}_{j)}$ vanishes—if the gradient tensor of $\boldsymbol{\mathcal{V}}$ is totally antisymmetric, which is a defining characteristic of a rigid frame (98)—then the derivatives in (133) are *separately* metric compatible. In this case $\boldsymbol{\mathcal{V}}$ satisfies *Killing's equation*

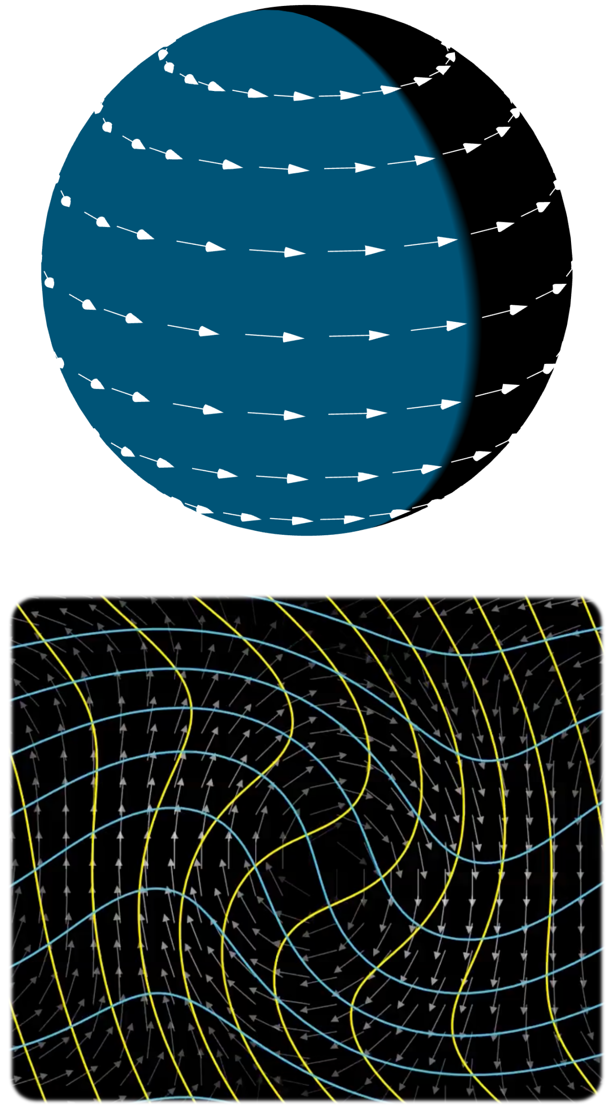

FIG. 2. The setting velocity $\boldsymbol{\mathcal{V}}$, shown in white, is a vector field that generates a diffeomorphism of space. **Top**: $\boldsymbol{\mathcal{V}} = \boldsymbol{\Omega} \times \boldsymbol{r}$ generates rotation of the sphere. In this case, $\boldsymbol{\mathcal{V}}$ is a Killing vector field because rotation is a rigid transformation for which $\nabla_{(i}\mathcal{V}_{j)} = 0$. **Bottom**: A swirling and shearing diffeomorphism of the plane. Here $\nabla_{(i}\mathcal{V}_{j)} \neq 0$, and we see a straining deformation. There is no divergent deformation because this $\boldsymbol{\mathcal{V}}$ was generated from a stream function, so $\nabla_i \mathcal{V}^i = 0$.

$$\mathcal{L}_{\mathcal{V}} g_{ij} = 0, \tag{137}$$

(Carroll 2004) equation (B29), and the fantastical force vanishes. The setting velocity $\boldsymbol{\mathcal{V}}$ then generates a rigid transformation of space if and only if it is a Killing vector field. A Killing field is a vector field representing an isometry (a symmetry that preserves distances), often described as a vector field that produces a conserved quantity like energy or momentum in a space-time. See §3.8 and Appendix B of Carroll (2004) for a discussion of Killing vectors and symmetries of the metric tensor.

7. Conclusions

In this paper, we have developed a tensor-based framework for describing geophysical fluid dynamics in general, deforming coordinate systems. Beginning from first principles in tensor analysis, we introduced a generalized intrinsic material derivative that correctly captures the evolution of acceleration tensor components in curvilinear bases that vary arbitrarily, but continuously, in space and time. This derivative ensures consistency across covariant and contravariant representations and across inertial and non-inertial frames, providing a unified treatment of the fundamental building blocks of GFD.

The generalized intrinsic time derivative $\delta/\delta t|^{\chi}$, taking its most general form in equation (127), allows us to calculate the total derivatives of arbitrary, space-time dependent tensor fields along a parameterized path $\chi(t)$, which is one path traveling along an arbitrary vector field $\boldsymbol{q}$, in coordinates being deformed by an arbitrary vector field $\boldsymbol{\mathcal{V}}$. If we set $\boldsymbol{\mathcal{V}} = 0$, the generalized derivative reduces to the intrinsic time derivative discussed by Grinfeld (2013) and Aris (1962). The only constraint on $\boldsymbol{\mathcal{V}}$ is that it generate an invertible, continuously differentiable transformation of the moving coordinates x^i—that is, that it generate a *diffeomorphism* on Euclidean space. In the particular case where $\boldsymbol{q} = \boldsymbol{v}$, the fluid velocity, the generalized intrinsic derivative reduces to the generalized intrinsic material derivative $\delta/\delta t$.

All coordinate systems (or settings as we call them to emphasize that unlike frames they are also rotating, accelerating, and deforming) typically used in GFD–rotating, accelerating, density, terrain-following, pressure, tracer, geodetic, geocentric, spherical, cylindrical–are a diffeomorphism away from an inertial, Cartesian frame. Taking into account the implications of such a coordinate transformation, we have found the general form for the apparent forces that correct for the non-inertial aspects of these coordinate systems without approximation.

We applied this generalized intrinsic derivative and derived explicit expressions for all apparent accelerations arising in GFD settings. These include the familiar fictitious forces–Coriolis, centrifugal, translational, and Euler forces–as well as the additional family of terms that arise when the coordinate system is deforming, which we have termed the *fantastical* forces. Unlike fictitious forces, the fantastical forces depend on the symmetric part of the setting-velocity gradient and therefore vanish only when the setting velocity is a Killing field, i.e., for diffeomorphisms connecting between rigid frames. Their dependence on tensor valence highlights that deformation introduces qualitatively new geometric structure into the apparent dynamics.

Our results clarify the relationship between classical rotating-frame dynamics and their fully general counterparts, showing that the standard GFD equations emerge as a special case of a broader tensorial formulation. By expressing all results using index notation, we provide equations that can be directly translated into numerical model implementations, particularly those employing vertical remapping, adaptive, or arbitrarily Lagrangian-Eulerian (ALE) coordinate systems.

The following statements are all logically equivalent:

1. The moving setting x is a rigid frame,
2. $\boldsymbol{\nabla} \otimes \boldsymbol{\mathcal{V}}$ is antisymmetric,
3. $\mathcal{L}_{\boldsymbol{\mathcal{V}}} g_{ij} = 0$
4. $\left.\frac{\partial g_{ij}}{\partial t}\right|_x = 0$,
5. $\boldsymbol{\mathcal{V}}$ is a Killing vector field,
6. Oldroyd's upper and lower convected derivatives are equivalent,
7. The fantastical forces are zero.

In many GFD settings, these statements do not apply, and thus nonzero fantastical forces will arise. Although they may be small, they are the agents of preservation of exact symmetries and conservation under all coordinate changes.

The framework developed here lays the mathematical foundation for the companion papers of this series, which will extend these ideas to the full oceanic primitive equations (including coordinate-independent specifications of common parameterizations); detail common coordinate systems of GFD in a tensorial framework; provide coordinate-independent averaging and coarse-graining operators; provide an asymptotic scaling for the fantastical forces (and other novelties); and derive tensorial budgets for common higher-order invariants (vorticity, potential vorticity, energy). Together, these works aim to establish a unified tensorial approach as a basis for next-generation ocean models, enabling more accurate representations of the physics of rotating, stratified flows in complex, time-varying geometries.

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Data availability statement. No data was used in this paper.

APPENDIX A

The metrinilic property for Jacobians

In this appendix we show that the metrinilic property of the covariant derivative extends to its action on Jacobians. To begin with we need to establish that the covariant derivative is meaningfully able to operate on the basis vectors. The covariant derivative of the Kronecker delta function is

$$\nabla_i \delta^j_k = 0 = \nabla_i \left(\boldsymbol{e}^j \cdot \boldsymbol{e}_k \right) \tag{A1}$$

which from the form of the covariant derivative applied to a mixed-valence second-order tensor (31) becomes

$$\begin{aligned}\nabla_i \delta^j_k = \boldsymbol{e}^j \cdot \partial_i \boldsymbol{e}_k + \boldsymbol{e}_k \cdot \partial_i \boldsymbol{e}^j \\ + \Gamma^j{}_{i\ell} \left(\boldsymbol{e}^\ell \cdot \boldsymbol{e}_k \right) - \Gamma^\ell{}_{ik} \left(\boldsymbol{e}^j \cdot \boldsymbol{e}_\ell \right) = 0\end{aligned} \tag{A2}$$

Gathering terms, we have

$$\boldsymbol{e}^j \cdot \left(\partial_i \boldsymbol{e}_k - \Gamma^\ell{}_{ik} \boldsymbol{e}_\ell \right) + \boldsymbol{e}_k \cdot \left(\partial_i \boldsymbol{e}^j + \Gamma^j{}_{i\ell} \boldsymbol{e}^\ell \right) = 0 \tag{A3}$$

and if we demand that the product rule (32) is satisfied, we also have

$$\boldsymbol{e}^j \cdot \nabla_i \boldsymbol{e}_k + \boldsymbol{e}_k \cdot \nabla_i \boldsymbol{e}^j = 0. \tag{A4}$$

This indicates that the action of the covariant derivative on the basis vectors is given by

$$\nabla_i \boldsymbol{e}_j = \partial_i \boldsymbol{e}_j - \Gamma^k{}_{ij} \boldsymbol{e}_k = 0 \tag{A5}$$

$$\nabla_i \boldsymbol{e}^j = \partial_i \boldsymbol{e}^j + \Gamma^j{}_{ik} \boldsymbol{e}^k = 0 \tag{A6}$$

which are seen to both vanish on account of (4).

With this established, we now turn to evaluating the covariant derivative of the Jacobian coefficients. We find

$$\nabla_i J^j_a = \nabla_i \left(\boldsymbol{e}^j \cdot \boldsymbol{e}_a \right) = \boldsymbol{e}^j \cdot \nabla_i \boldsymbol{e}_a. \tag{A7}$$

$$\nabla_i \boldsymbol{e}_a = \partial_i \boldsymbol{e}_a - \Gamma^k{}_{ia} \boldsymbol{e}_k \tag{A8}$$

We then have for $\nabla_i J^j_a$

$$0 = \nabla_i \delta^j_k = \nabla_i \left(J^j_a J^a_k \right) = J^a_k \nabla_i J^j_a + J^j_a \cancelto{0}{\nabla_i J^a_k} = J^a_k \nabla_i J^j_a \tag{A9}$$

from which it follows that $\nabla_i J^j_a$ itself vanishes. The derivations of $\nabla_a J^i_b = 0$ and $\nabla_a J^b_i = 0$ are exactly the same.

a. Lie derivatives of Jacobians

We can follow a similar procedure to define the Lie derivatives of the Jacobians. Beginning in the same way, we use the fact that the Lie derivative of the Kronecker delta is zero, and we demand that the Lie derivative satisfy the product rule across dot products

$$\begin{aligned} 0 &= \mathcal{L}_{\boldsymbol{\mathcal{V}}} \delta^i_j \\ &= \mathcal{V}^k \cancelto{0}{\nabla_k \delta^i_j} - \delta^k_j \nabla_k \mathcal{V}^i + \delta^i_k \nabla_j \mathcal{V}^k \\ &= -\boldsymbol{e}_j \cdot \boldsymbol{e}^k \nabla_k \mathcal{V}^i + \boldsymbol{e}^i \cdot \boldsymbol{e}_k \nabla_j \mathcal{V}^k \end{aligned} \tag{A10}$$

and demanding that the Lie derivative satisfy the product rule

$$\boldsymbol{e}_j \cdot \mathcal{L}_{\mathcal{V}} \boldsymbol{e}^i + \boldsymbol{e}^i \cdot \mathcal{L}_{\mathcal{V}} \boldsymbol{e}_j \tag{A11}$$

indicates that the action of the Lie derivative on the basis vectors is given by

$$\mathcal{L}_{\mathcal{V}} \boldsymbol{e}_i = \boldsymbol{e}_j \nabla_i \mathcal{V}^j \tag{A12}$$

$$\mathcal{L}_{\mathcal{V}} \boldsymbol{e}^i = -\boldsymbol{e}^j \nabla_j \mathcal{V}^i \tag{A13}$$

which, as with the covariant derivative, is exactly what we would have found had we naïvely applied the Lie derivative based on index placement. We can now calculate the Lie derivative of the Jacobians using the product rule:

$$\begin{aligned} \mathcal{L}_{\mathcal{V}} J^i_a &= \mathcal{L}_{\mathcal{V}} \boldsymbol{e}^i \cdot \boldsymbol{e}_a \\ &= \boldsymbol{e}_a \cdot \mathcal{L}_{\mathcal{V}} \boldsymbol{e}^i + \boldsymbol{e}^i \cdot \mathcal{L}_{\mathcal{V}} \boldsymbol{e}_a \\ &= -\boldsymbol{e}_a \cdot \boldsymbol{e}^j \nabla_j \mathcal{V}^i + \boldsymbol{e}^i \cdot \boldsymbol{e}_b \nabla_a \mathcal{V}^b \\ &= -J^j_a \nabla_j \mathcal{V}^i + J^i_b \nabla_a \mathcal{V}^b \\ &= -\nabla_a \mathcal{V}^i + \nabla_a \mathcal{V}^i \\ &= 0 \end{aligned} \tag{A14}$$

Derivatives of all useful metric-related objects are compiled in Table A1. Space-time derivative commutators as applied to vector components appear in Table A2.

APPENDIX B

The intrinsic derivative for covariant components

In this appendix we explicitly derive the form of the intrinsic derivative when applied to the covariant components of a vector w_i. The partial derivative $\partial_i \boldsymbol{w}$ can be rewritten as

$$\partial_i \boldsymbol{w} = \boldsymbol{e}_j \nabla_i w^j, \qquad \nabla_i w^j \equiv \partial_i w^j + \Gamma^j{}_{ik} w^k \tag{30}$$

$$\partial_i \boldsymbol{w} = \boldsymbol{e}^j \nabla_i w_j, \qquad \nabla_i w_j \equiv \partial_i w_j - \Gamma^k{}_{ij} w_k \tag{B1}$$

using (5). Thus the covariant derivative ∇_i, viewed as an operator, takes on a different form when applied to contravariant and covariant components of a vector. The derivative of a vector along a curve χ can be written as

$$\left. \frac{d\boldsymbol{w}}{dt} \right|^{\chi} = \left. \frac{\delta w_a}{\delta t} \right|^{\chi} \boldsymbol{e}^a \tag{B2}$$

TABLE A1. A table of derivatives of the metric coefficients.

Derivative	δ^i_j	δ^a_b	g_{ij}	g_{ab}	$\boldsymbol{e}_i$	$\boldsymbol{e}^i$	$\boldsymbol{e}_a$	$\boldsymbol{e}^a$	J^i_a	J^a_i
∂_k	0	0	$2\Gamma_{(i,j)k}$	$2\Gamma_{(a,b)c}J^c_k$	$\boldsymbol{e}_j\Gamma^j{}_{ik}$	$-\boldsymbol{e}^j\Gamma^i{}_{jk}$	$\boldsymbol{e}_b\Gamma^b{}_{ac}J^c_k$	$-\boldsymbol{e}^b\Gamma^a{}_{bc}J^c_k$	$J^i_{ac}J^c_k$	J^a_{ik}
∂_c	0	0	$2\Gamma_{(i,j)k}J^k_c$	$2\Gamma_{(a,b)c}$	$\boldsymbol{e}^j\Gamma^j{}_{ik}J^k_c$	$-\boldsymbol{e}^j\Gamma^i{}_{jk}J^k_c$	$\boldsymbol{e}_b\Gamma^b{}_{ac}$	$-\boldsymbol{e}^b\Gamma^a{}_{bc}$	J^i_{ac}	$J^a_{ik}J^k_c$
$\frac{\partial}{\partial t}\big\|_x$	0	0	$2\nabla_{(i}\mathcal{V}_{j)}$	$\mathcal{V}^c\partial_c g_{ab}$	$\boldsymbol{e}_j\nabla_i\mathcal{V}^j$	$-\boldsymbol{e}^j\nabla_j\mathcal{V}^i$	$\boldsymbol{e}_c\Gamma^c{}_{ab}\mathcal{V}^b$	$-\boldsymbol{e}^b\Gamma^a{}_{bc}\mathcal{V}^c$	$-J^i_b\partial_a\mathcal{V}^b$	$J^b_i\partial_b\mathcal{V}^a$
$\frac{\partial}{\partial t}\big\|_y$	0	0	$2\partial_{(i}\mathcal{V}_{j)}$	0	$\boldsymbol{e}_j\partial_i\mathcal{V}^j$	$-\boldsymbol{e}^j\partial_j\mathcal{V}^i$	0	0	$-J^j_a\partial_j\mathcal{V}^i$	$J^a_j\partial_i\mathcal{V}^j$
∇_k	0	0	0	0	0	0	0	0	0	0
∇_c	0	0	0	0	0	0	0	0	0	0
$\mathcal{L}_\mathcal{V}$	0	0	$2\nabla_{(i}\mathcal{V}_{j)}$	$2\nabla_{(a}\mathcal{V}_{b)}$	$\boldsymbol{e}_j\nabla_i\mathcal{V}^j$	$-\boldsymbol{e}^j\nabla_j\mathcal{V}^i$	$\boldsymbol{e}_b\nabla_a\mathcal{V}^b$	$-\boldsymbol{e}^b\nabla_b\mathcal{V}^a$	0	0
$\frac{\delta}{\delta t}\big\|^\chi$	0	0	0	0	0	0	0	0	0	0
$\frac{\delta}{\delta t}$	0	0	0	0	0	0	0	0	0	0

TABLE A2. A table of derivative commutators. The commutator is understood to be applied to the relevant components of the tensor in each column.

Commutator	$\boldsymbol{w}$
$[\frac{\partial}{\partial t}\big\|_x, \partial_j]$	0
$[\frac{\partial}{\partial t}\big\|_x, \nabla_j]$	$w^k\nabla_k\nabla_j\mathcal{V}^i$
$[\mathcal{L}_\mathcal{V}, \nabla_j]$	$w^k\nabla_k\nabla_j\mathcal{V}^i$
$[\frac{\delta}{\delta t}\big\|^\chi, \nabla_j]$	$-(\nabla_j q^k)(\nabla_k w^i)$
$[\frac{\delta}{\delta t}, \nabla_j]$	$-(\nabla_j v^k)(\nabla_k w^i)$
$\left[\frac{\partial}{\partial t}\big\|_y, \partial_b\right]$	0
$\left[\frac{\partial}{\partial t}\big\|_y, \nabla_b\right]$	0
$[\mathcal{L}_\mathcal{V}, \nabla_b]$	$w^c\nabla_c\nabla_b\mathcal{V}^a$
$[\frac{\delta}{\delta t}\big\|^\chi, \nabla_b]$	$-(\nabla_b q^c)(\nabla_c w^a)$
$[\frac{\delta}{\delta t}, \nabla_b]$	$-(\nabla_b v^c)(\nabla_c w^a)$

where the intrinsic derivative of the covariant components,

$$\frac{\delta w_a}{\delta t}\bigg|^\chi \equiv \frac{\partial w_a}{\partial t}\bigg|_y + q^b\nabla_b w_a, \tag{B3}$$

has the same form as the intrinsic derivative of the contravariant components presented earlier in (56). This is because all dependence on the valence of tensor components is already accounted for by the covariant derivative.

To proceed, we will need expressions for the temporal rates of change of the reciprocal basis vectors $\boldsymbol{e}^i$. One finds

$$\frac{\partial \boldsymbol{e}_i}{\partial t}\bigg|_x = \boldsymbol{e}_j\nabla_i\mathcal{V}^j, \qquad \frac{\partial \boldsymbol{e}^i}{\partial t}\bigg|_x = -\boldsymbol{e}^j\nabla_j\mathcal{V}^i \tag{B4}$$

the first identity of which was presented earlier in (68). To derive the second identity, we note

$$\frac{\partial}{\partial t}\left(\boldsymbol{e}^i\cdot\boldsymbol{e}_j\right)\bigg|_x = \frac{\partial\delta^i_j}{\partial t}\bigg|_x = 0 \tag{B5}$$

and consequently, from the product rule,

$$\boldsymbol{e}_j\cdot\frac{\partial \boldsymbol{e}^i}{\partial t}\bigg|_x = -\boldsymbol{e}^i\cdot\frac{\partial \boldsymbol{e}_j}{\partial t}\bigg|_x. \tag{B6}$$

Combining this with (68) yields

$$\boldsymbol{e}_j\cdot\frac{\partial \boldsymbol{e}^i}{\partial t}\bigg|_x = -\boldsymbol{e}^i\cdot\boldsymbol{e}_k\nabla_j\mathcal{V}^k = -\nabla_j\mathcal{V}^i \tag{B7}$$

which is equivalent to the second identity in (B4), as both give $-\nabla_j\mathcal{V}^i$ for the jth covariant component of $\partial\boldsymbol{e}^i/\partial t\big|_x$.

The derivative of the vector $\boldsymbol{w}$ along the curve χ in the moving setting, expressed in terms of the vector's covariant components, is then

$$\left.\frac{d\boldsymbol{w}}{dt}\right|^{\mathcal{X}} = \left.\frac{\partial w_i}{\partial t}\right|_x \boldsymbol{e}^i + \left(p^j \nabla_j w_i\right) \boldsymbol{e}^i + \boxed{\left. w_i \frac{\partial \boldsymbol{e}^i}{\partial t}\right|_x} \tag{B8}$$

which is to be compared with (70) in terms of the contravariant components. Again the boxed term is the new term that arises in comparison with the fixed coordinate system. From (B4) we then obtain

$$\left.\frac{\delta w_i}{\delta t}\right|^{\mathcal{X}} \equiv \left.\frac{\partial w_i}{\partial t}\right|_x + p^j \nabla_j w_i - \boxed{w_j \nabla_i \mathcal{V}^j} \tag{73}$$

as the covariant version of (71), in comparison to which the final boxed term appears with opposite sign and with a different arrangement of indices. The material version (73) follows.

We have pointed out that, despite the fact that (71) and (73) give the contravariant and covariant components of the same vector, the same is true neither the first nor third terms on the right-hand sides considered separately. In fact on account of $\partial g_{ij}/\partial t|_x = 2\nabla_{(i}\mathcal{V}_{j)}$, we have

$$\left.\frac{\partial w_i}{\partial t}\right|_x - g_{ij} \left.\frac{\partial w^j}{\partial t}\right|_x = 2 w^j \nabla_{(i}\mathcal{V}_{j)} \tag{B9}$$

by the product rule, after recognizing that $w_i = g_{ij} w^j$. The partial time derivatives $\partial w_i/\partial t|_x$ and $\partial w^i/\partial t|_x$ are clearly not the components of the same vector.

It is thus instructive to attempt to obtain (73) by lowering the index on the contravariant version. Beginning with

$$\left.\frac{\delta w^i}{\delta t}\right|^{\mathcal{X}} \equiv \left.\frac{\partial w^i}{\partial t}\right|_x + p^j \nabla_j w^i + w^j \nabla_j \mathcal{V}^i \tag{71}$$

we lower the i index through a contraction with g_{ij}, yielding

$$\left.\frac{\delta w_i}{\delta t}\right|^{\mathcal{X}} = g_{ij} \left.\frac{\partial w^j}{\partial t}\right|_x + p^j \nabla_j w_i + w^j \nabla_j \mathcal{V}_i. \tag{B10}$$

Substituting the result (B9) then leads to

$$\left.\frac{\delta w_i}{\delta t}\right|^{\mathcal{X}} = \left.\frac{\partial w_i}{\partial t}\right|_x + p^j \nabla_j w_i + w^j \left[\nabla_j \mathcal{V}_i - 2\nabla_{(j}\mathcal{V}_{i)}\right] \tag{B11}$$

using the symmetry of $\nabla_{(i}\mathcal{V}_{j)}$ to reverse the i and j indices. A tensor minus twice its symmetric portion is equal to the negative of its own transpose. This yields

$$\left.\frac{\delta w_i}{\delta t}\right|^{\mathcal{X}} \equiv \left.\frac{\partial w_i}{\partial t}\right|_x + p^j \nabla_j w_i - w^j \nabla_i \mathcal{V}_j \tag{B12}$$

which matches (73) after swapping valences on w^j and V_j.

APPENDIX C

The apparent rate of change of the setting velocity

In this appendix we derive an equation for the apparent rate change of the setting velocity $\boldsymbol{\mathcal{V}}$ at a fixed location in the moving setting,

$$\left.\frac{\partial \mathcal{V}^i}{\partial t}\right|_x \boldsymbol{e}_i = \frac{d\boldsymbol{\Omega}}{dt} \times \boldsymbol{r} + \frac{d^2 \boldsymbol{S}}{dt^2} - \boldsymbol{\Omega} \times \frac{d\boldsymbol{S}}{dt}. \tag{C1}$$

The total derivative of the setting velocity $\boldsymbol{\mathcal{V}}$ at fixed x location is, from the product rule

$$\left.\frac{d\boldsymbol{\mathcal{V}}}{dt}\right|_x = \left.\frac{\partial \mathcal{V}^i}{\partial t}\right|_x \boldsymbol{e}_i + \mathcal{V}^i \left.\frac{\partial \boldsymbol{e}_i}{\partial t}\right|_x \tag{C2}$$

which rearranges to give

$$\left.\frac{\partial \mathcal{V}^i}{\partial t}\right|_x \boldsymbol{e}_i = \left.\frac{d\boldsymbol{\mathcal{V}}}{dt}\right|_x - (\boldsymbol{\mathcal{V}} \cdot \boldsymbol{\nabla})\, \boldsymbol{\mathcal{V}} \tag{C3}$$

using (68) for the derivative of the basis vectors. The two terms on the right-hand side are found to be, respectively,

$$\left.\frac{d\boldsymbol{\mathcal{V}}}{dt}\right|_x = \frac{d\boldsymbol{\Omega}}{dt} \times \boldsymbol{r} + \boldsymbol{\Omega} \times \left.\frac{d\boldsymbol{r}}{dt}\right|_x + \frac{d^2\boldsymbol{S}}{dt^2} \tag{C4}$$

$$(\boldsymbol{\mathcal{V}} \cdot \boldsymbol{\nabla})\, \boldsymbol{\mathcal{V}} = \boldsymbol{\Omega} \times \left(\boldsymbol{\Omega} \times \boldsymbol{r} + \frac{d\boldsymbol{S}}{dt}\right) \tag{C5}$$

where the latter equality follows from the advection identity in the case of rigid motion, $\boldsymbol{\mathcal{V}} = \boldsymbol{\Omega} \times \boldsymbol{r} + d\boldsymbol{S}/dt$. For rigid motion we have

$$\left.\frac{d\boldsymbol{r}}{dt}\right|_x = \boldsymbol{\Omega} \times \boldsymbol{r} \tag{C6}$$

and thus (C4) and (C5) subtract to yield (C1), as claimed.

APPENDIX D

Intrinsic Derivative of Reynolds Stress $u^i u^j$

First we note that the Lie derivative has the following properties:

$$\mathcal{L}_{\boldsymbol{A}+\boldsymbol{B}} w^i = \mathcal{L}_{\boldsymbol{A}} w^i + \mathcal{L}_{\boldsymbol{B}} w^i \tag{D1}$$

$$\mathcal{L}_{\boldsymbol{w}} w^i = 0 \tag{D2}$$

Then we have

$$\mathcal{L}_{\boldsymbol{\mathcal{V}}} v^i = -\mathcal{L}_{\boldsymbol{u}} v^i \tag{D3}$$

$$\mathcal{L}_{\boldsymbol{\mathcal{V}}} u^i = \mathcal{L}_{\boldsymbol{v}} u^i \tag{D4}$$

and the intrinsic derivative of u^i becomes

$$\begin{aligned}\frac{\delta u^i}{\delta t} &= \left.\frac{\partial u^i}{\partial t}\right|_x + v^k \nabla_k u^i - \mathcal{L}_{\boldsymbol{v}} u^i \\ &= \left.\frac{\partial u^i}{\partial t}\right|_x + u^k \nabla_k v^i\end{aligned} \tag{D5}$$

Because the intrinsic derivative obeys the product rule, we then find the derivative of the Reynolds stress to be

$$\begin{aligned}\frac{\delta\left(u^i u^j\right)}{\delta t} &= \frac{\delta u^i}{\delta t} u^j + u^i \frac{\delta u^j}{\delta t} \\ &= \left.\frac{\partial\left(u^i u^j\right)}{\partial t}\right|_x + u^i u^k \nabla_k v^j + u^j u^k \nabla_k v^i\end{aligned} \tag{D6}$$

APPENDIX E

Intrinsic Derivative of Scalar Field τ

The intrinsic material derivative of a scalar field is particularly simple:

$$\begin{aligned}\frac{\delta \tau}{\delta t} &= \left.\frac{\partial \tau}{\partial t}\right|_y + v^b \nabla_b \tau \\ &= \left(\left.\frac{\partial}{\partial t}\right|_y + v^b \partial_b\right) \tau\end{aligned} \tag{E1}$$

and similarly from the moving, deforming perspective

$$\begin{aligned}\frac{\delta \tau}{\delta t} &= \left.\frac{\partial \tau}{\partial t}\right|_x + v^k \nabla_k \tau - \mathcal{L}_{\mathcal{V}} \tau \\ &= \left.\frac{\partial \tau}{\partial t}\right|_x + v^k \partial_k \tau - \mathcal{V}^k \partial_k \tau \\ &= \left(\left.\frac{\partial}{\partial t}\right|_x + u^k \partial_k\right) \tau\end{aligned} \tag{E2}$$

Thus, there are no equivalents to the apparent forces to be applied for scalar field evolution.

APPENDIX F

Oldroyd and the Intrinsic Time Derivative

The generalized intrinsic derivative lets us easily derive the expression for the convected derivative of Oldroyd (1950). Oldroyd is concerned with materially advected coordinates, and so in the following we choose the specific $\boldsymbol{\mathcal{V}} = \boldsymbol{v}$. In the spirit of the rest of this paper, we prefer his original, indicial definition. This definition is preferable also because it makes clear that the upper and lower convected derivatives are not, in fact, two different derivatives of tensors, but the same derivative applied respectively to contravariant and covariant tensor *components*, in precisely the same sense as the Lie derivative or the covariant derivative. For brevity's sake, we show the derivation for the derivative as applied to vector components, however it generalizes immediately to tensors of any order. As described in the paragraph preceding Oldroyd's equation (23), the convected derivative of a vector $\boldsymbol{w} = w^i \boldsymbol{e}_i$ is nothing but the vector with components $\partial w^i / \partial t|_x$ in the moving x-setting. The question is: what are the inertial y-frame components of this vector? We treat these as tensor components by assumption, and so they transform with the Jacobian:

$$\begin{aligned}\frac{\mathfrak{d} w^a}{\mathfrak{d} t} &\equiv J_i^a \left.\frac{\partial w^i}{\partial t}\right|_x \\ &= J_i^a \left(\frac{\delta w^i}{\delta t} - v^k \nabla_k w^i + \mathcal{L}_{\boldsymbol{v}} w^i\right) \\ &= \frac{\delta w^a}{\delta t} - v^c \nabla_c w^a + \mathcal{L}_{\boldsymbol{v}} w^a \\ &= \left.\frac{\partial w^a}{\partial t}\right|_y + \mathcal{L}_{\boldsymbol{v}} w^a\end{aligned} \tag{F1}$$

And there we have it – we have recovered Oldroyd's equation (23)[8]. In operator form, the convected derivative is therefore

$$\frac{\mathfrak{d}}{\mathfrak{d} t} = \left(\frac{\delta}{\delta t} - \nabla_{\boldsymbol{v}} + \mathcal{L}_{\boldsymbol{v}}\right) \tag{F2}$$

and this can be applied either to y-frame components:

$$\frac{\mathfrak{d} w^a}{\mathfrak{d} t} = \left(\frac{\delta}{\delta t} - v^c \nabla_c + \mathcal{L}_{\boldsymbol{v}}\right) w^a = \left(\left.\frac{\partial}{\partial t}\right|_y + \mathcal{L}_{\boldsymbol{v}}\right) w^a \tag{F3}$$

or to x-setting components:

$$\frac{\mathfrak{d} w^i}{\mathfrak{d} t} = \left(\frac{\delta}{\delta t} - v^k \nabla_k + \mathcal{L}_{\boldsymbol{v}}\right) w^i = \left.\frac{\partial w^i}{\partial t}\right|_x \tag{F4}$$

Later, coordinate-free definitions of the convected derivative, for example in Hinch and Harlen (2021), are unnatural because even though they are superficially coordinate-free, they nevertheless depend on the valence of the underlying tensor components. The "upper convected" and "lower convected" derivatives can respectively be written in terms of Oldroyd's component-wise derivative $\mathfrak{d}/\mathfrak{d}t$ like so:

$$\begin{aligned}\overset{\triangledown}{\boldsymbol{w}} &= \boldsymbol{e}_a \frac{\mathfrak{d} w^a}{\mathfrak{d} t} = \boldsymbol{e}_i \frac{\mathfrak{d} w^i}{\mathfrak{d} t} \\ \overset{\vartriangle}{\boldsymbol{w}} &= \boldsymbol{e}^a \frac{\mathfrak{d} w_a}{\mathfrak{d} t} = \boldsymbol{e}^i \frac{\mathfrak{d} w_i}{\mathfrak{d} t}\end{aligned} \tag{F5}$$

[8] His formulation includes extra terms to handle the differentiation of tensor densities, all of which are zero for tensors.

Notice the similarity in form here to equations (60) and (72). This definition generalizes readily to higher-order tensors of any indicial signature, including mixed component representations. These two quantities are not equal because the Lie derivative, and therefore also Oldroyd's, acts on the metric coefficients to produce a non-zero tensor:

$$\frac{\mathfrak{d} g_{ab}}{\mathfrak{d} t} = \cancelto{0}{\left.\frac{\partial g_{ab}}{\partial t}\right|_y} + \mathcal{L}_{\boldsymbol{v}} g_{ab} = 2\nabla_{(a} v_{b)} \tag{F6}$$

and so we find that

$$\overset{\triangle}{\boldsymbol{w}} - \overset{\triangledown}{\boldsymbol{w}} = 2\boldsymbol{e}^a w^b \nabla_{(a} v_{b)} = \boldsymbol{e}^a w^b \mathcal{L}_{\boldsymbol{v}} g_{ab} \tag{F7}$$

So these derivatives coincide if and only if $\boldsymbol{v}$ is a Killing field, which would correspond physically to a rigidly translating and rotating fluid – a very restrictive constraint.

We have shown, for the particular case $\mathcal{V} = \boldsymbol{v}$, that the generalized intrinsic derivative and Oldroyd's derivative encode similar information about time-varying tensors, but they are nearly opposite in character.

Oldroyd's derivative begins with the components $\partial w^i \partial t|_x$ and asserts that, when transformed as tensor components, these look a certain way in the y^a coordinates. The convected derivative is therefore constructed to produce the second form from the first, and, for vector fields, provides the coordinates of the vector field $\partial \boldsymbol{w}/\partial t|_y + [\boldsymbol{v}, \boldsymbol{w}]$, where the brackets indicate the commutator operation.

In contrast, the generalized intrinsic derivative begins with the tensor field $d\boldsymbol{w}/dt$, and asks: what are the components of this in each of the bases $\boldsymbol{e}_a$ and $\boldsymbol{e}_i$. The derivative is therefore constructed to act on either set of components w^a or w^i to return the components of $d\boldsymbol{w}/dt$ in the relevant basis.

We prefer the latter conceptual framing because the relevant coordinate-free tensor field $d\boldsymbol{w}/dt$ is easier to interpret than the vector field $\partial \boldsymbol{w}/\partial t|_y + [\boldsymbol{v}, \boldsymbol{w}]$.

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