Fantastical Forces and Where to Find Them or: A Tensor Time Derivative for Geophysical Fluids in Deforming Coordinate Systems

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ABSTRACT: We begin the development of a tensor-based formalism for geophysical fluid dynamics using differential geometry to express the oceanic primitive equations in time-varying, curvilinear coordinates. We first extend the definition of the intrinsic time derivative on tensor components to accommodate arbitrarily time-varying coordinate systems. This allows us to make rigorous the transformation from inertial to accelerating reference frames, systematically deriving apparent forces and sorting them into two classes: the familiar "fictitious" forces arising from rigid transformations such as rotation, and the unfamiliar "fantastical" forces arising from non-rigid, deforming transformations. Though many coordinate choices in ocean modeling are deforming, these latter apparent forces are often overlooked. This work and companion papers provide a unifying mathematical foundation that improves communication among ocean modelers, permits higher-order, multi-dimensional numerical schemes, and forms a basis for the development of next-generation ocean models and parameterizations.

SIGNIFICANCE STATEMENT: The purpose of this work is to clarify the effects of moving coordinate systems on ocean models. Ocean modelers often choose coordinate systems that move along with the water to simplify computation, but just as rotation introduces Coriolis and centrifugal forces, general time-dependence in model coordinates introduces apparent forces that have not been fully described. To describe these forces rigorously, we first extend an existing method of time-differentiation to account for the movement of coordinate systems themselves and then apply it to the fluid velocity field to derive an entirely general fluid momentum equation. We discuss the characteristics of these forces, compare them to familiar analogs, and classify them based on symmetry.

1. Introduction

Geophysical fluid dynamics (GFD) is a non-relativistic branch of fluid dynamics in the Newtonian limit of gravity, centered on the effects of stratification, rotation, and topography. These features break the isotropy of fluid motion, privileging flow, stirring, and mixing in certain directions and within particular curved surfaces, for example isosurfaces of neutral buoyancy. Numerical models exploit the geometry of broken symmetries with dynamically motivated choices of coordinates, such as a vertical direction aligned with effective gravity and thus hydrostatic balance. Such coordinate choices simplify equations and increase computational efficiency.

Coordinate systems do not exist in nature; this is precisely what makes them so useful because they can be freely chosen to match the geometry of the physical problem at hand. Nature prescribes the quantities—density, momentum, strain—that determine the state of a system, along with the laws by which the state evolves, and these physical quantities themselves are coordinate-free. This freedom from coordinate dependence is what we mean when we say that such quantities are tensor fields: all choices of their representations—all sets of components—correspond to the same object in nature. Technical definitions of tensors as transforming in certain ways under a change of coordinates or as multilinear functions from vectors to scalars are all constructed to ensure that we maintain a sharp separation between the tensor itself and our representation of that tensor's components.

Working with tensors in generalized coordinates, we can on the one hand choose any representation of nature that suits our problem, and on the other hand ensure that we are dealing with real, physical quantities that are not simply artifacts of our coordinate choice. Tensor equations that apply under all coordinates are particularly useful in geophysical fluid dynamics where we use space and time-varying coordinates to capture planetary rotation, and often also choose time-varying coordinates to simplify the dynamics of stratification.

Tensor analysis has the power to express the equations of fluid motion without approximation in as general a coordinate system as possible. This includes familiar, rigidly rotating and translating coordinate systems as well as potentially deforming systems such as isopycnal or terrain-following coordinates. Our first goal is to define a component-wise time derivative $\delta/\delta t$ which, when applied to the components of a tensor $\boldsymbol{T}$ in some basis, provides the components of the time derivative of the tensor $d\boldsymbol{T}/dt$ in

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the same basis. Such an operation must permit both spatial and temporal dependence of the coordinate bases linked to those components. The *intrinsic time derivative*, discussed in Aris (1962) and Grinfeld (2013) accomplishes this goal for spatially varying basis vectors, however, its definition presumes that for a given location the basis is fixed in time; this is virtually never the case in geophysical fluid dynamics. The *generalized intrinsic time derivative* introduced here extends this idea to permit time-dependent basis vectors. It takes on its most general form in section 4h. The specific form of this derivative for contravariant vector components along a material path, our equation (82), appears in Luo and Bewley (2004) as equation (23). We generalize the derivative to act on tensors of any order or valence.

Changes observed in the tensor field $\boldsymbol{T}$ depend not only on physical processes, but also on the observer's path through it, and the generalized intrinsic time derivative captures both types of change. If the observer's location tracks a fluid parcel, as in the Lagrangian fluid equations, this intrinsic derivative reduces to an intrinsic *material* derivative. With this material derivative finally in hand, we proceed to give expressions for the components of inertial velocity and acceleration in all coordinate systems, including rigidly rotating and translating but also deforming systems. These expressions include the familiar advection terms and "fictitious forces"—the Coriolis and centrifugal forces—but also include other, unfamiliar apparent forces that arise only in deforming systems. With the choice of Lagrangian coordinates, we connect these new terms to the convected time derivative of Oldroyd (1950) and to the Lie derivative that is so useful in geometric mechanics (Holm 2025).

In GFD, different temporally and spatially dependent coordinate systems, often quite complex, are chosen for convenience in different situations. Most fundamentally, rotating frames of reference are typically used to separate the large velocities of planetary motion from the relative velocities more easily measured by earthbound observers. Similarly, deforming coordinates are often used to capture the anisotropic flows along the most natural surfaces.

In physical oceanography, coordinate choices generally depend on the region of interest (Fox-Kemper et al. 2019): near the surface and in regions of weak stratification, height coordinates are used; near the sea floor, topography drives dynamics and terrain-following, or σ, coordinates are preferred; and away from the boundaries in regions of strong stratification, isopycnal coordinates (McDougall 1987; Young 2012) are typically chosen to capture the undulation of density surfaces. More generally, Lagrangian coordinates (e.g., Salmon 2013) flow with a fluid while Eulerian systems stay fixed in coordinate space as the fluid moves through them. The Arbitrary Lagrangian-Eulerian method (e.g., Petersen et al. 2015; Gibson et al. 2017; Megann et al. 2022) and other complex coordinates as used in the HYCOM (Bleck 2002; Halliwell 2004) and MOM6 (Griffies et al. 2020) numerical models combine these approaches, allowing the grid to flow with the fluid for a time, and resetting before it becomes too deformed.

The choice of coordinate system bears upon the accuracy of numerical models. Finite machine precision can introduce error into simulations, and an improper choice of coordinates will exacerbate that error. For example, is has long been known that modeling tracer transport in height coordinates leads to the Veronis effect: if lateral transport is calculated perpendicular to the geopotential gradient at constant height, rather than more realistically along surfaces of constant density, the misalignment of these two sorts of surfaces causes spurious mixing across density classes (Veronis 1975). The difference of roughly eight orders of magnitude between isopycnal and diapycnal mixing in the real ocean means that small errors in the representation of isopycnal surfaces introduce significant spurious diapycnal mixing—if even one one-millionth of the isopycnal mixing is erroneously rotated into the diapycnal direction, it will overwhelm the true diapycnal mixing. More suitable choices of coordinates can help to avoid this (Griffies et al. 2000, 2015; Ilıcak et al. 2012).

Coordinate-free tensor formulations find application in numerical methods modeling fluid flows for other reasons as well. An important class of totally anti-symmetric tensors, *differential forms*, are especially useful for integration and discretization, particularly on unstructured meshes (Crane et al. 2013; Crane 2018), and Cotter and Thuburn (2014) have applied these methods to the rotating shallow water equations. Åhlander et al. (2001), meanwhile, note that "the design of applications for scientific computing should be based firstly upon continuous abstractions and secondly on approximations," and show that separating software engineering concerns by the level of mathematical abstraction allows discretization to occur more naturally at each level. The literature on the Sophus software library details many of these discrete, tensor-based methods (Grant et al. 2000; Engø et al. 2001; Friis et al. 2001; Haveraaen et al. 2005; Åhlander and Otto 2006; Haveraaen and Friis 2009; Heinzl and Schwaha 2011; Gilbert and Vanneste 2023).

There is plenty of precedent for the application of tensor analysis to fluid mechanics in other domains, most obviously in relativistic fluids where spacetime itself—not just the coordinate system—is so curved as to require careful treatment. The black hole accretion study by Hawley et al. (1984) is exemplary. It is therefore natural to introduce a time derivative to non-relativistic dynamics by taking the limit $v \ll c$, where $c \approx 3 \times 10^8$ m s^{-1} is the speed of light. Maddison and Marshall (2013) take this approach to deriving the Eliassen-Palm flux tensor in time-varying coordinates. However, this introduces a speed-of-light velocity scale c that is cumbersome alongside atmospheric winds and ocean currents with mean velocities six to ten

orders of magnitude smaller. We avoid this by taking a non-relativistic approach from the start, which has the added benefit of grounding the problem in familiar, flat 3D space.

Putting aside relativistic physics, introductory fluids texts (e.g., Kundu et al. 2016) often frame equations in Cartesian tensor notation with a fixed basis $\{\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}\}$. Crucially for our purposes, partial derivatives of tensor components with respect to spatial coordinates do not immediately generalize to deforming, curvilinear coordinates, but require the added algebraic structure of the *covariant derivative*, discussed subsequently. To fully generalize component-wise tensor computations to non-orthogonal coordinates, components must be tracked with respect to both a coordinate basis $\boldsymbol{e}_i$ and its dual or reciprocal basis $\boldsymbol{e}^i$, which happen to coincide in Cartesian coordinates; a considerable simplification.

This work builds on the rich history of advances in tensor analysis in fluid mechanics. Tensor analysis for geophysical fluids is reviewed in Griffies (2004, Chapter 18) and has been applied to the Gent-McWilliams skew flux (Griffies 1998; Dukowicz and Smith 1997) and to thickness-weighted averaging (Young 2012). Oldroyd (1950) employs tensor analysis to define time derivatives for materially advected coordinates, although in quite a different sense than we do, and Hinch and Harlen (2021) provide relevant context for Oldroyd's work. Grinfeld (2009) employs tensor analysis in a fully curvilinear treatment of thin fluid films, which Grinfeld (2013) expands into the full calculus of moving surfaces. This latter text is the inspiration for the extrinsic perspective on differential geometry that we assume here, and Grinfeld's equation (16.27) for the invariant time derivative is a sort of analog to our generalization of the intrinsic derivative, but restricted to moving, embedded surfaces. Thiffeault (2001) derived an abstract form for covariant time derivatives $\mathcal{D}X^a \equiv \dot{X}^a + \alpha^a{}_b X^b$, varying the definition of the *connexion* $\alpha^a{}_b$ to recover Oldroyd's convective derivative, the corotational Jaumann derivative, and the directional derivative. Some other choice of connexion might recover the intrinsic derivative discussed here.

The goal of this work is three-fold. Firstly, we aim to create a primer on the tensor analysis of Euclidean space in curvilinear coordinates, to highlight the remarkable power and compactness of index notation, and to provide tensor component-based equations that can be directly translated into model code. This approach offers high levels of both abstraction and practicality, since any expression written in index notation is immediately computable. Secondly, we construct the generalized intrinsic time derivative discussed above. Finally, we use this derivative to investigate the apparent forces that arise in the general, non-inertial, deforming coordinate systems so common in GFD. These apparent forces separate cleanly into two categories based on tensor symmetries: the familiar fictitious forces arising from the rigid (translational and/or rotational) part of the coordinate transformation, and what we dub the *fantastical forces* arising from the non-rigid (straining and/or diverging) part. The intrinsic derivative and the resulting momentum equation provide a generalized method for the use of time-varying coordinates in three dimensions, and generalize earlier methods that deform only in the vertical, for example the treatment of the dia-surface velocity by Griffies et al. (2020).

The structure of this paper is as follows: sections 2 and 3 lay out the definitions, notation, and conventions that we employ for the tensor calculus of Euclidean space; section 4 derives the form of the generalized intrinsic time derivative; section 5 applies it to the momentum equation, recovers the familiar rotating equations of motion from the general form, discusses generalizations of the standard apparent forces, and relates this to related work in solid mechanics; and section 6 discusses the vorticity equation.

2. Preliminaries: definitions, notation, and conventions

This section introduces the definitions, notation, and conventions of tensor analysis employed throughout. Prior familiarity with tensor analysis is assumed; readers already proficient in tensor analysis using index notation can skip to section 4 where the GFD-specific aspects begin. For extensive treatments of this topic, the authors favor Grinfeld (2013), Carroll (2004), Itskov (2015), Aris (1962), and Ogden (1984); see also Lilly et al. (2024). For brevity's sake, many statements found readily in the literature are given without proof.

One of the goals of this work, and of this section specifically, is to be a brief introductory reference for tensor analysis and differential geometry as needed for fluid applications, in particular numerical modeling of the ocean circulation in moving coordinate systems. This leads us to make two fundamental choices: *extrinsic geometry*, meaning that the existence of flat, three-dimensional Euclidean, inertial space may be presumed, and *explicit notation*, in which we use index-based notation for tensor analysis so that mathematical expressions can be readily converted into computer code for numerical calculations. Our presentation aims not only to introduce useful abstractions but also to eschew needless ones. We occasionally refer back to equations in their symbolic form when index notation becomes onerous to interpret.

a. Space, coordinate systems, and basis vectors

In non-relativistic three-dimensional Euclidean space, the setting for GFD, time is a parameter shared by all spatial locations. A *scalar field* is a distribution of values at each point in space. A *vector field* can be visualized as a distribution of arrows, having magnitude and direction, at each point in space. Scalar and vector fields respectively are tensor fields of order zero and one. A *second-order tensor field* is a linear mapping from vectors to vectors, or

a bilinear mapping from pairs of vectors to scalars, at each point in space. More generally, an nth order tensor field is a multilinear function from n vectors to scalars, although in GFD we rarely encounter tensors higher than second order. The terms "scalar," "vector," and "tensor" are understood implicitly to refer to fields rather than single values, and therefore always depend on space and time.

A *coordinate system* is a set of enumerated surfaces extending throughout space, not necessarily planar nor spatially fixed in time, that can be used to specify locations and therefore to quantify motion. These coordinate surfaces are indexed with a set of three numbers $\{x^1, x^2, x^3\}$ called the *coordinates*. When used as the argument to a function, the shorthand notation $f(x)$ will be used in place of $f(x^1, x^2, x^3)$, following Grinfeld (2013). The *position vector field* $\boldsymbol{r}(x)$ of the coordinate system is the vector field that at each coordinate location $\{x^1, x^2, x^3\}$ gives the displacement vector from the origin to that location. For notational compactness, the argument x will generally be omitted from $\boldsymbol{r}$ and other tensor fields. We shall consider x^i to refer collectively to the entire set $\{x^1, x^2, x^3\}$, and similarly for other indexed entities.

Critically, physical laws and the tensors appearing in them are not affected by the choice of coordinate systems. However, the representation of tensors as sets of components and the representation of physical laws as equations expressed in terms of these tensor components *are* affected by the choice of coordinate system. Much of the effort expended here is in deriving equations for tensor *components*, because these component equations are what we actually use for calculation in numerical models. In our experience the distinction between tensors and their components is frequently confounded in GFD, particularly when taking material derivatives, which are often applied to components alone rather than full tensors.

Differentiation of the position vector field $\boldsymbol{r}$ with respect to each coordinate defines a set of *basis vectors*

$$\boldsymbol{e}_i \equiv \partial_i \boldsymbol{r} \,, \tag{1}$$

where $\partial_i \equiv \partial / \partial x^i$ is a shorthand for the partial derivative with respect to the ith coordinate. A basis vector is thus the tangent vector to the curve traced out by the change in position as we vary $\boldsymbol{r}$ one coordinate x^i at a time. The basis vectors need not have unit length, be orthogonal to one another, or be constant in space or time. Immediately we see the influence of extrinsic geometry here: the vector $\boldsymbol{r}$ exists independent of our representation of space as a differentiable manifold. Were we strictly intrinsic, we would have to make the more abstract choice $\boldsymbol{e}_i = \partial_i$ for the basis. Singularities in the coordinate systems can be managed, for example at extrema in non-monotonic coordinates and at poles (see, e.g., Prusa 2018), so long as there are not too many. So, coordinates imply a basis, and in casual exposition we may freely refer to either.

In order to handle so many possibilities for the basis $\boldsymbol{e}_j$, a set of *reciprocal basis vectors* $\boldsymbol{e}^i$ are defined such that

$$\boldsymbol{e}^i \cdot \boldsymbol{e}_j \equiv \delta^i_j \,, \tag{2}$$

with δ^i_j being the Kronecker delta function

$$\delta^i_j \equiv \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases} \tag{3}$$

Unlike in Cartesian coordinates, the $\boldsymbol{e}_i$ and $\boldsymbol{e}^i$ bases are not generally orthonormal themselves. Equation (2) introduces a new type of joint orthonormality.

Due to the definition in equation (1), the basis vectors $\boldsymbol{e}_i$ will shorten when coordinates are packed more tightly – that is, they "covary" with the coordinates. The basis vectors $\boldsymbol{e}^i$ contravary with the coordinates in just such a way to make equation (2) hold. This justifies the names *covariant* and *contravariant*, and motivates the subscripting of covariant indices and the superscripting of contravariant indices.

Components of tensor fields also covary or contravary with the coordinates. For those familiar with tensors in Cartesian coordinates, but not general curvilinear coordinates, the distinction between these two types of components is a key new concept with critical implications for deforming coordinates.

In what follows we will make use of the partial derivatives of the basis and reciprocal basis vectors with respect to the coordinates. In flat space, these can be expressed as

$$\partial_i \boldsymbol{e}_j = \Gamma^k{}_{ij} \boldsymbol{e}_k \,, \qquad \partial_i \boldsymbol{e}^j = -\Gamma^j{}_{ik} \boldsymbol{e}^k \,, \tag{4}$$

where $\Gamma^k{}_{ij}$, the *Christoffel symbol of the second kind*, is defined as

$$\Gamma^k{}_{ij} \equiv \left(\partial_i \boldsymbol{e}_j\right) \cdot \boldsymbol{e}^k. \tag{5}$$

Thus $\Gamma^k{}_{ij}$ is the expansion coefficient for the derivative of the jth basis vector with respect to the ith coordinate that is associated with the kth basis vector. One may readily show the symmetry $\Gamma^k{}_{ij} = \Gamma^k{}_{ji}$ from (1) and the commutativity of partial derivatives. A readily computable expression for the Christoffel symbols will be presented shortly.

The extrinsic approach to tensor analysis offers important simplifications over the intrinsic approach common in general relativity. There, definitions may reference only the curved spacetime manifold because an extrinsic space is not observable. With an extrinsic space, however, we can think of basis vectors $\boldsymbol{e}_i$ simply as spatially and temporally varying arrows in physical space rather than abstractly as differential operators in the tangent space of the spacetime manifold. Furthermore, here the basis vectors $\boldsymbol{e}_i$ and reciprocal basis vectors $\boldsymbol{e}^i$ are elements of the same Euclidean vector space, whereas in general relativity the reciprocal

basis $\boldsymbol{e}^i$ must be replaced with the dual basis $\mathrm{d}x^i$ of differential 1-forms in the cotangent space to the manifold. See section 2.4 of Carroll (2004) or section 1.4 of Ogden (1984) for further discussion.

The general, time-varying, curvilinear coordinate system x^i captures both the rotational acceleration and deformation required for GFD. In addition, we introduce a second, potentially curvilinear but rigid and inertial coordinate system y^a. On account of Galilean invariance we are free to choose an absolute reference for motion that defines this inertial coordinate system to be stationary.

A sketch is presented in Fig. 1, which also includes notation introduced subsequently. In order to clearly distinguish between tensor components referenced to these two systems, we adopt the convention of indexing any object as referenced from the non-inertial, deforming setting with the letters $\{i, j, k, \ell, ...\}$ and from the inertial frame with $\{a, b, c, d, ...\}$. The less common choices of letters in the latter case are intended to reflect the less common choice of an inertial coordinate system for GFD.

Let $\boldsymbol{s}$ be the position vector field associated with the y^a coordinate system, and $\boldsymbol{S} = \boldsymbol{s} - \boldsymbol{r}$ be the location of origin O_x of the x^i coordinates with respect to the origin O_y of the y^a system; see Figure 1. The y^a system has covariant basis vectors

$$\boldsymbol{e}_a \equiv \partial_a \boldsymbol{s} = \partial_a \boldsymbol{r} \,, \tag{6}$$

where $\partial_a \equiv \partial/\partial y^a$, and where the latter equality follows from the fact that $\boldsymbol{S}$ is spatially constant. Note that the basis vectors $\boldsymbol{e}_i$ and $\boldsymbol{e}_a$ are distinguished by their indices, as are the partial derivatives ∂_i and ∂_a, the Christoffel symbols $\Gamma^k{}_{ij}$ and $\Gamma^c{}_{ab}$, and so forth. While the curvilinear coordinates in the inertial y^a system may have complex spatial dependence, the time derivatives of these coordinates, their basis vectors, and any other quantities such as $\boldsymbol{s}$ and Christoffel symbols are all temporally constant. In the x^i system, the corresponding quantities may all vary in time.

b. Index notation

The extrinsic perspective allows a vector $\boldsymbol{w}$ to be equivalently represented in terms of the basis or the reciprocal basis as

$$\boldsymbol{w} = w^i \boldsymbol{e}_i = w_i \boldsymbol{e}^i \,, \tag{7}$$

where (2) is used to find the components as $w^i \equiv \boldsymbol{w} \cdot \boldsymbol{e}^i$ and $w_i \equiv \boldsymbol{w} \cdot \boldsymbol{e}_i$, called the *contravariant* and *covariant* components of $\boldsymbol{w}$, respectively. Note that we use the Einstein summation convention: any indices occurring in one upper and one lower position are understood to be summed over, so that $w^i \boldsymbol{e}_i$ means $\sum_{i=1}^{3} w^i \boldsymbol{e}_i$. For the special case of a Cartesian basis, the covariant and contravariant basis vectors are identical, and consequently so are the covariant and contravariant components of $\boldsymbol{w}$. In curvilinear coordinates, however, the index position is essential to keep track of which set of components is being considered.

Similarly, a second-order tensor $\boldsymbol{A}$ can be variously represented in terms of its contravariant, covariant, or mixed components as

$$\begin{aligned} \boldsymbol{A} = A^{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j = A_{ij} \boldsymbol{e}^i \otimes \boldsymbol{e}^j \\ = A^i{}_j \boldsymbol{e}_i \otimes \boldsymbol{e}^j = A_i{}^j \boldsymbol{e}^i \otimes \boldsymbol{e}_j, \end{aligned} \tag{8}$$

where $\otimes$ denotes the tensor product. Each of these expressions involves a sum over nine elements, with two nested sums over three basis vectors in each sum. As with vector components, the tensor components can be found by projecting $\boldsymbol{A}$ onto the basis vectors, $A^{ij} \equiv \boldsymbol{e}^i \cdot \left(\boldsymbol{A}\boldsymbol{e}^j\right)$ and so forth, again using orthogonality (2) together with the defining property

$$(\boldsymbol{a} \otimes \boldsymbol{b})\, \boldsymbol{w} = \boldsymbol{a}\, (\boldsymbol{b} \cdot \boldsymbol{w}) \tag{9}$$

of the tensor product (see Itskov 2015, section 1.7).

The index state of a set of tensor components—its number of indices together with their status as contravariant or covariant—is known as its *valence* (e.g., Needham 2021, §33.1). Unlike the vector components w^i, the coordinates x^i are *not* the components of some vector field $\boldsymbol{x}$ as the notation appears to imply; they are just the numbers specifying the location of a particular point with respect to the chosen coordinate system. In particular, they are not the same as the components r^i of the position vector field $\boldsymbol{r}$, except in Cartesian coordinates.

We adopt the convention that the left-to-right order of the indices in expressions such as $A^{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$ must match between the components and the basis. With this convention, it becomes unnecessary to explicitly indicate the basis vectors that accompany any tensor's components, since their arrangement is evident from placement of the indices within the components. That is, the tensor component A^{ij} may be used to unambiguously refer to the entire tensor $A^{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$. Consequently, one need not write the basis vectors at all in evaluating tensor operations, just the relations among components. However, by the product rule the derivatives of tensors depend on both the derivatives of the components and the derivatives of the basis vectors. It follows that in order to dispense entirely with basis vectors some special machinery for expressing derivatives in terms of tensor components is needed, as introduced shortly.

c. Index juggling

The basis vectors and reciprocal basis vectors can be used to define the *metric coefficients*

$$g_{ij} \equiv \boldsymbol{e}_i \cdot \boldsymbol{e}_j, \qquad g^{ij} = \boldsymbol{e}^i \cdot \boldsymbol{e}^j \tag{10}$$

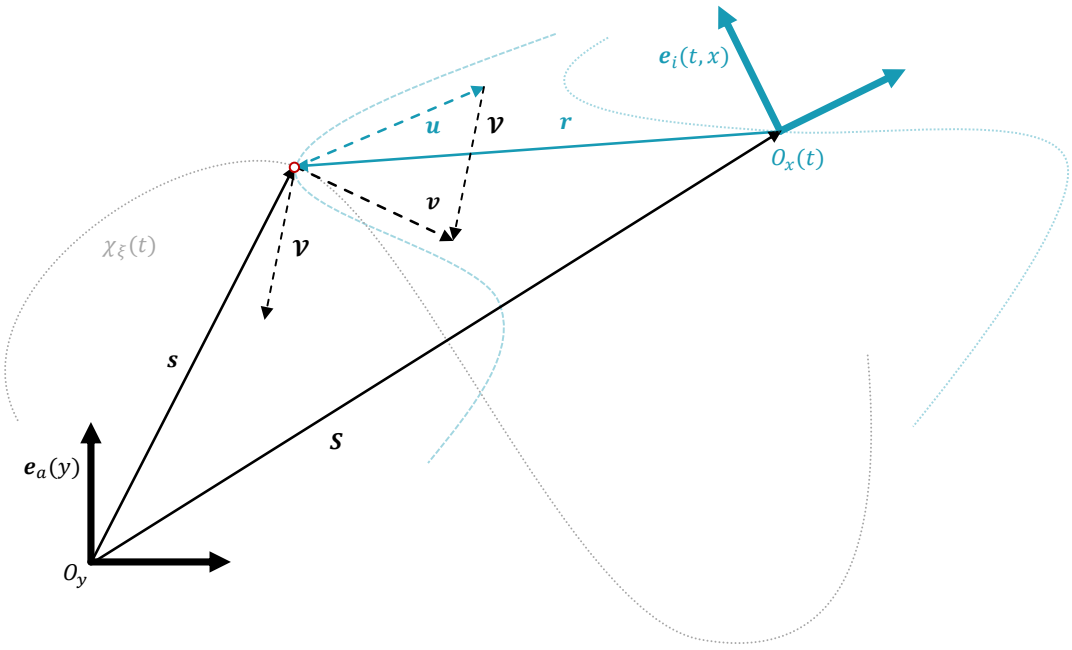


FIG. 1. A schematic showing quantities in an inertial frame (black), chosen without loss of generality to be stationary, and a general moving, deforming coordinate system as often used in GFD (teal). Heavy solid lines indicate coordinate axes, thin solid lines are position vectors, thin dashed lines are velocity vectors, and dashed or dotted curves are trajectories. The inertial coordinate system $y^a = \{y^1, y^2, y^3\}$ has basis vectors $\boldsymbol{e}_a$ and fixed origin O_y, while the moving coordinate system $x^i = \{x^1, x^2, x^3\}$ has basis vectors $\boldsymbol{e}_i$ and moving origin $O_x(t)$, the path of which is indicated by the dotted teal curve. The instantaneous displacement at some time t_0 from the fixed origin O_y to the moving origin $O_x(t)$ is given by the vector $\boldsymbol{S}$. Addition information on this plot is not used until later sections. Then, we will consider a parameterized path $\chi_\xi(t)$, representing the time-varying position of a fluid parcel labeled ξ, the position of which at t_0 is indicated by the small white circle. The position vectors of this parcel relative to O_y and $O_x(t)$ at time t_0 are $\boldsymbol{s}$ and $\boldsymbol{r}$ respectively, which satisfy $\boldsymbol{s} = \boldsymbol{r} + \boldsymbol{S}$. At time t_0, the fluid parcel has velocity $\boldsymbol{v}$ as observed from the inertial y coordinate system and velocity $\boldsymbol{u}$ as observed from the moving, deforming x coordinate system. Through a combination of the motion of $O_x(t)$ and the internal motion and deformation of the x coordinate system, the x coordinate location occupied by the fluid parcel also traces out a curve in space, indicated by the dashed teal curve. At time t_0, the velocity of this x coordinate location as viewed from the inertial frame is the vector $\boldsymbol{\mathcal{V}}$, which is necessarily tangent to the curve. Note that $\boldsymbol{e}_i$ is a function of time *and space*, so each location in space has a potentially different basis vector set. The velocities are related by $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{\mathcal{V}}$, as shown. While we have illustrated this with an individual parcel, $\boldsymbol{v}$, $\boldsymbol{u}$, and $\boldsymbol{\mathcal{V}}$ are vector fields which add in this way at each point in the fluid.

which are symmetric and are not equal to δ^i_j except in the special case of orthonormal basis sets. With these definitions, the contravariant and covariant components of the vector $\boldsymbol{w} = w^i \boldsymbol{e}_i = w_i \boldsymbol{e}^i$ can converted into one another. This conversion proceeds as

$$w_j g^{ji} = w_j \boldsymbol{e}^j \cdot \boldsymbol{e}^i = \boldsymbol{w} \cdot \boldsymbol{e}^i = w^j \boldsymbol{e}_j \cdot \boldsymbol{e}^i = w^j \delta^i_j = w^i \quad (11)$$

$$w^j g_{ji} = w^j \boldsymbol{e}_j \cdot \boldsymbol{e}_i = \boldsymbol{w} \cdot \boldsymbol{e}_i = w_j \boldsymbol{e}^j \cdot \boldsymbol{e}_i = w_j \delta^j_i = w_i \quad (12)$$

which together establish that

$$g^{ij} w_j = w^i, \qquad g_{ij} w^j = w_i. \quad (13)$$

Using the metric coefficients to convert between covariant and contravariant components is called *index raising and lowering*, *index juggling*, or *changing valence*. Note that index raising inverts index lowering, and thus

$$g^{ij} g_{jk} \equiv \delta^i_k, \quad (14)$$

which traditionally defines g^{ij}. The second of equations (10) follows from the first together with (14).

d. Tensor operations

When performing operations on tensors, we have the choice of working in symbolic notation with familiar operators such as $\times$, $\otimes$, and $\boldsymbol{\nabla}$, or in index notation where we represent the impact of these operation on tensor *components*. In Tables 1–3, we present translations between these two notations for common operations and identities. See section 1 of Itskov (2015) or Lilly et al. (2024) for the symbolic definitions of the tensor product $\boldsymbol{a} \otimes \boldsymbol{b}$, left operation $\boldsymbol{a}\boldsymbol{A}$, the tensor transpose $\boldsymbol{A}^T$, and the tensor trace $\mathrm{tr}(\boldsymbol{A})$,

TABLE 1. Common vector operations, written both in symbolic notation and in index notation, with the basis vectors associated with the index notation expressions shown in the final column. Here $\boldsymbol{\nabla} \equiv \boldsymbol{e}^i \partial_i$ denotes the familiar nabla or del operator as expressed in curvilinear coordinates, $\otimes$ denotes the tensor product, ∇_i is the covariant derivative acting on tensor components, and $\boldsymbol{\varepsilon}^{ijk}$ is the Levi-Civita or alternating tensor, all of which are defined subsequently. In later tables we will omit the final column because the basis is implied by index placement.

Name	Symbolic	Index	Basis
Inner product	$\boldsymbol{a} \cdot \boldsymbol{b}$	$a_i b^i$	—
Cross product	$\boldsymbol{a} \times \boldsymbol{b}$	$\boldsymbol{\varepsilon}^{ijk} a_j b_k$	$\boldsymbol{e}_i$
Tensor product	$\boldsymbol{a} \otimes \boldsymbol{b}$	$a^i b_j$	$\boldsymbol{e}_i \otimes \boldsymbol{e}^j$
Gradient	$\boldsymbol{\nabla} \gamma$	$\nabla_i \gamma$	$\boldsymbol{e}^i$
Divergence	$\boldsymbol{\nabla} \cdot \boldsymbol{w}$	$\nabla_i w^i$	—
Curl	$\boldsymbol{\nabla} \times \boldsymbol{w}$	$\boldsymbol{\varepsilon}^{ijk} \nabla_j w_k$	$\boldsymbol{e}_i$
Gradient tensor	$\boldsymbol{\nabla} \otimes \boldsymbol{w}$	$\nabla_i w^j$	$\boldsymbol{e}^i \otimes \boldsymbol{e}_j$

all of which may be defined without coordinates. Note that because $\boldsymbol{a} \cdot (\boldsymbol{A}\boldsymbol{b}) = (\boldsymbol{a}\boldsymbol{A}) \cdot \boldsymbol{b}$, we may write simply $\boldsymbol{a}\boldsymbol{A}\boldsymbol{b}$.

TABLE 2. Common vector identities involving the tensor product, written in both symbolic and index notation.

Name	Symbolic	Symbolic	Index
Tensor trace	$\mathrm{tr}\,(\boldsymbol{a} \otimes \boldsymbol{b})$	$\boldsymbol{a} \cdot \boldsymbol{b}$	$a_i b^i$
Right operation	$(\boldsymbol{a} \otimes \boldsymbol{b})\,\boldsymbol{w}$	$\boldsymbol{a}\,(\boldsymbol{b} \cdot \boldsymbol{w})$	$a^i b_j w^j$
Left operation	$\boldsymbol{w}\,(\boldsymbol{a} \otimes \boldsymbol{b})$	$(\boldsymbol{w} \cdot \boldsymbol{a})\,\boldsymbol{b}$	$w_i a^i b_j$
Transpose	$(\boldsymbol{a} \otimes \boldsymbol{b})^T$	$\boldsymbol{b} \otimes \boldsymbol{a}$	$b^i a_j$
Advection	$\boldsymbol{u}\,(\boldsymbol{\nabla} \otimes \boldsymbol{w})$	$(\boldsymbol{u} \cdot \boldsymbol{\nabla})\,\boldsymbol{w}$	$u^j \nabla_j w^i$
—	$\boldsymbol{u}\,(\boldsymbol{\nabla} \otimes \boldsymbol{w})^T$	$(\boldsymbol{\nabla} \otimes \boldsymbol{w})\,\boldsymbol{u}$	$u^j \nabla^i w_j$

The action of the tensor $\boldsymbol{A}$ on the vector $\boldsymbol{w}$ is represented as

$$\boldsymbol{A}\boldsymbol{w} = A^i{}_j \left(\boldsymbol{e}_i \otimes \boldsymbol{e}^j\right) w^k \boldsymbol{e}_k = A^i{}_j w^j \boldsymbol{e}_i \tag{15}$$

again using (9) and (2). But because the presence of the $\boldsymbol{e}_i$ can be inferred, we can write down $A^i{}_j w^j$ as an implicit representation of $\boldsymbol{A}\boldsymbol{w}$ without needing to carry out the intervening manipulations on the basis vectors. Matching an upper with a lower index in two tensor representations, as in $A^i{}_j w^j$, effects an inner product and is called *contraction*. Each contraction involves the implicit dot product of a basis vector $\boldsymbol{e}_i$ and a reciprocal basis vector $\boldsymbol{e}^j$ using (2), reducing the degree of the tensor by two. Index notation readily accommodates multiple simultaneous contractions, easily identified by matching indices.

A particularly important tensor is the *identity tensor*, which in flat Euclidean space is given by

$$\boldsymbol{I} = g^{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j = g_{ij} \boldsymbol{e}^i \otimes \boldsymbol{e}^j = \delta^i_j \boldsymbol{e}_i \otimes \boldsymbol{e}^j = \delta^j_i \boldsymbol{e}^i \otimes \boldsymbol{e}_j . \tag{16}$$

Its non-mixed components are the metric coefficients, while its mixed components are Kronecker delta functions. Given a flat, extrinsic space, g_{ij}, g^{ij}, and δ^i_j are all representations of $\boldsymbol{I}$ in precisely the same sense as the various representations of $\boldsymbol{A}$ in equation (8), and the metric $\boldsymbol{g}$ is nothing but the identity tensor $\boldsymbol{I}$. This tensor has the property that $\boldsymbol{I}\boldsymbol{w} = \boldsymbol{w}$ for all $\boldsymbol{w}$. The identity tensor is implicitly involved whenever the delta function is used in index notation, that is, $a_i \delta^i_j b^j = a_i b^i$ corresponds to the tensor equation $\boldsymbol{a}\boldsymbol{I}\boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{b}$ with $\boldsymbol{I}$ chosen in a mixed representation. It is also implicitly involved in index juggling, since $g^{ij} w_j = w^i$ corresponds to the tensor equation $\boldsymbol{I}\boldsymbol{w} = \boldsymbol{w}$ with $\boldsymbol{I}$ chosen in a non-mixed representation. This is intuitive, because changing the representation of tensor components is an operation that does not change the tensor itself, that is, an identity operation.

Evaluation of the cross product and curl in index notation requires the introduction of an appropriate tensor operator. The *Levi-Civita symbol* ϵ_{ijk}, given by the rule

$$\epsilon_{ijk} = \epsilon^{ijk} \equiv \begin{cases} 1, & ijk \text{ an even permutation of } 123 \\ -1, & ijk \text{ an odd permutation of } 123 \\ 0, & \text{otherwise} \end{cases} \tag{17}$$

is used in turn to define the components of the *Levi-Civita tensor* $\boldsymbol{\varepsilon}$, also known as the *volume element*. Its covariant representation is

$$\boldsymbol{\varepsilon}_{ijk} \equiv \sqrt{g} \epsilon_{ijk} , \tag{18}$$

where $g \equiv |g_{ij}|$ is the determinant of the metric components. With orthonormal basis vectors we have $g = 1$. As the components of a tensor, the Levi-Civita's tensor indices may be raised in the usual way, and the contravariant components of $\boldsymbol{\varepsilon}$ are

$$\boldsymbol{\varepsilon}^{ijk} = \frac{1}{\sqrt{g}} \epsilon^{ijk} . \tag{19}$$

The contraction of these two representations satisfies the useful identity

$$\boldsymbol{\varepsilon}_{ijk} \boldsymbol{\varepsilon}^{k\ell m} = \delta^\ell_i \delta^m_j - \delta^m_i \delta^\ell_j \tag{20}$$

which can be used to easily prove most of the triple-product identities that must be memorized or looked up for manipulation of symbolic equations.

Notation will be needed for isolating portions of a second-order tensor $\boldsymbol{A}$ exhibiting various types of symmetries, presented in Table 3. In symbolic notation, a

TABLE 3. Operations for isolating the portions of a second-order tensor $\boldsymbol{A}$ exhibiting various types of symmetries in symbols and index notation.

Part	Symbolic	Index
Isotropic	$\frac{1}{3}\mathrm{tr}(\boldsymbol{A})\boldsymbol{I}$	$\frac{1}{3}g_{ij}A^{k}{}_{k}$
Symmetric	$\frac{1}{2}\left(\boldsymbol{A}+\boldsymbol{A}^{T}\right)$	$\frac{1}{2}\left(A_{ij}+A_{ji}\right)\equiv A_{(ij)}$
Skew-symmetric	$\frac{1}{2}\left(\boldsymbol{A}-\boldsymbol{A}^{T}\right)$	$\frac{1}{2}\left(A_{ij}-A_{ji}\right)\equiv A_{[ij]}$

tensor is said to *symmetric* if $\boldsymbol{A}=\boldsymbol{A}^{T}$ and *antisymmetric* or *skew-symmetric* if $\boldsymbol{A}=-\boldsymbol{A}^{T}$. In index notation, a tensor is symmetric if

$$A_{ij}=A_{ji}, \qquad A^{i}{}_{j}=A_{j}{}^{i}, \qquad A^{ij}=A^{ji} \tag{21}$$

and skew symmetric if

$$A_{ij}=-A_{ji}, \qquad A^{i}{}_{j}=-A_{j}{}^{i}, \qquad A^{ij}=-A^{ji}. \tag{22}$$

Using index juggling, all statements in (21) and (22) are shown to be true if any one of them is true. Note that for a symmetric tensor $\boldsymbol{A}$, there is no ambiguity in writing the mixed components as $A^{i}_{j}=A^{i}{}_{j}=A_{j}{}^{i}$, without the placeholder spaces; this is the case for the Kronecker delta δ^{i}_{j}.

This discussion has shown that explicitly writing the basis vectors is superfluous because all relevant information is already present in the placement of indices. However, rather than identify w^{i} and $A^{i}{}_{j}$ with a vector and second-order tensor respectively, as is sometimes done, we find it helpful to emphasize that w^{i} and $A^{i}{}_{j}$ are tensor *components* which only when paired with basis vectors yield the actual tensors. This underscores the fact that derivatives of the tensors involve derivatives of both the tensor components *and* the basis vectors, even if the basis vectors are not explicitly written.

3. Spatial derivatives in curvilinear coordinates

We would like to express not only changes in tensor representations but also spatial and temporal differentiation in component form without explicit reference to the basis vectors. This requires that a derivative be expressed in such a way that it involves an action only on the components of the tensor and not on the basis vectors. It can be shown that partial derivatives of tensor components such as $\partial_{i}w^{j}$ and $\partial_{i}A^{j}{}_{k}$ do not transform according to the tensor transformation law (24) and are thus not themselves the components of tensors. An extension of partial derivatives that accounts for changing basis vectors is thus required to express the spatial derivatives of tensors in component form. This is precisely what the Christoffel symbols defined in equations (4) and (5) express.

Though our original definitions of the Christoffel symbols are geometrically instructive, the symbols can also be calculated more straight-forwardly in terms of the metric coefficients as

$$\Gamma^{i}{}_{jk}=\frac{1}{2}g^{i\ell}\left(\partial_{j}g_{\ell k}+\partial_{k}g_{\ell j}-\partial_{\ell}g_{jk}\right). \tag{23}$$

Under a change of basis, the components of a tensor transform according to the *tensor transformation law*

$$w^{a}=J^{a}_{i}w^{i}, \qquad w_{a}=J^{i}_{a}w_{i}, \qquad A^{a}{}_{b}=J^{a}_{i}J^{j}_{b}A^{i}{}_{j}, \tag{24}$$

where the *Jacobian coefficients* J^{i}_{a} and J^{a}_{i} are defined as

$$J^{a}_{i}\equiv\partial_{i}y^{a}, \qquad J^{i}_{a}\equiv\partial_{a}x^{i} \tag{25}$$

The transformation law (24) generalizes to tensors of any order, with a Jacobian contraction on each index. Note that in terms of the basis vectors, the Jacobian coefficients can also be written as[1]

$$J^{a}_{i}=\boldsymbol{e}_{i}\cdot\boldsymbol{e}^{a}, \qquad J^{i}_{a}=\boldsymbol{e}_{a}\cdot\boldsymbol{e}^{i}. \tag{26}$$

We also have, by the chain rule,

$$\boldsymbol{e}_{i}=J^{a}_{i}\boldsymbol{e}_{a}, \qquad \boldsymbol{e}_{a}=J^{i}_{a}\boldsymbol{e}_{i}, \tag{27}$$

meaning that J^{a}_{i} are the components of the $\boldsymbol{e}_{i}$ basis expanded in terms of the $\boldsymbol{e}_{a}$ basis, and conversely for J^{i}_{a}.

An indexed entity contains the components of a tensor *if and only if* it transforms according to (24). Indeed, this transformation law can be used as an alternate, equivalent definition of a tensor to the definition given above as a multilinear function on vectors, as shown in (1.6)–(1.10) of Jeevanjee (2011).

The Christoffel symbols $\Gamma^{k}{}_{ij}$ do not transform according to the tensor transformation law and are therefore not the components of a third-order tensor. A tensor with the same mixture of indices as $\Gamma^{k}{}_{ij}$ would transform according to

$$A^{a}{}_{bc}=J^{a}_{i}J^{j}_{b}J^{k}_{c}A^{i}{}_{jk}. \tag{28}$$

The Christoffel symbols, however, follow the transformation law

$$\Gamma^{a}{}_{bc}=J^{a}_{i}J^{j}_{b}J^{k}_{c}\Gamma^{i}{}_{jk}+J^{a}_{i}\frac{\partial^{2}x^{i}}{\partial y^{b}\partial y^{c}} \tag{29}$$

which differs from (28) in the second term. The reason the Christoffel symbols are not components of tensors is that

[1] To see this, note that the Jacobian J^{i}_{a} may be used to transform the basis vectors as $\boldsymbol{e}_{a}\equiv\partial_{a}\boldsymbol{s}=\partial_{a}\boldsymbol{r}=J^{i}_{a}\partial_{i}\boldsymbol{r}=J^{i}_{a}\boldsymbol{e}_{i}$. It then follows that $J^{a}_{i}=J^{a}_{j}\,\delta^{j}_{i}=J^{a}_{j}\left(\boldsymbol{e}_{i}\cdot\boldsymbol{e}^{j}\right)=\boldsymbol{e}_{i}\cdot J^{a}_{j}\boldsymbol{e}^{j}=\boldsymbol{e}_{i}\cdot\boldsymbol{e}^{a}$, and similarly for J^{i}_{a}.

$\Gamma^k{}_{ij}$, by construction according to (5), describes an aspect of the x^i coordinate system, namely how its basis vectors vary spatially. When we change coordinates to y^a, we are not simply changing the representation of the Christoffel symbols $\Gamma^k{}_{ij}$ but rather obtaining new quantities $\Gamma^a{}_{bc}$ that describe the spatial variations of a different set of basis vectors, the basis vectors $\boldsymbol{e}_a$ of the y^a system.

Nevertheless, the Christoffel symbols can be said to resemble tensors in the sense that the upper index and/or either one of the two lower indices will transform according to the transformation law, but both lower indices together will not. This can be understood using their definitions $\Gamma^k{}_{ij} \equiv (\partial_i \boldsymbol{e}_j) \cdot \boldsymbol{e}^k$ and $\Gamma^c{}_{ab} \equiv (\partial_a \boldsymbol{e}_b) \cdot \boldsymbol{e}^c$. The upper index k of $\Gamma^k{}_{ij}$ corresponds to the basis vectors we are expanding in; this index will therefore transform like the contravariant components of a tensor. The first lower index i corresponds to the coordinate system in which we are taking a partial derivative; this index will transform like the covariant components of a tensor. From the symmetry $\Gamma^k{}_{ij} = \Gamma^k{}_{ji}$, the second lower index can also be transformed like a tensor. Yet $\partial_i \boldsymbol{e}_j$ together does not transform like a tensor in j because this would require $\partial_i J^a_j = 0$, which is not the case in general; thus both lower indices do not simultaneously transform according to the tensor transformation law.

a. Changing representations involve the identity tensor

It builds intuition to highlight a detail that we have not seen discussed in the literature, namely the role of the identity tensor $\boldsymbol{I}$ in changing tensor representations. As discussed in section 2d, in Euclidean space the action of index juggling implicitly involves the identity tensor, since $g^{ij} w_j = w^i$ corresponds to the tensor equation $\boldsymbol{I}\boldsymbol{w} = \boldsymbol{w}$ with $\boldsymbol{I} = g^{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$. However, the identity tensor can also be written in the form

$$\boldsymbol{I} = J^a_i \boldsymbol{e}_a \otimes \boldsymbol{e}^i = J^i_a \boldsymbol{e}_i \otimes \boldsymbol{e}^a, \tag{30}$$

in which case the coefficients of the basis vectors are seen to be the Jacobian coefficients. In this form it is clear that changing representations from one basis to another also involves the identity tensor, since $w^a = J^a_i w^i$ again corresponds to $\boldsymbol{I}\boldsymbol{w} = \boldsymbol{w}$ but now with $\boldsymbol{I} = J^a_i \boldsymbol{e}_a \otimes \boldsymbol{e}^i$. In both cases the effect is to express the tensor in a new representation without changing the tensor itself. Moreover, combining (16) with (30), we also see that in Euclidean space, the Kronecker delta, the metric coefficients, and the Jacobian coefficients are all different representations of the identity tensor $\boldsymbol{I}$.

This clarifies several points of ambiguity in the literature. Commonly g_{ij} is referred to as the *metric tensor*, but from our extrinsic perspective, there is no separate "metric" tensor having the g_{ij} as its covariant components apart from the identity tensor $\boldsymbol{I} = g_{ij} \boldsymbol{e}^i \otimes \boldsymbol{e}^j$. For this reason we refer to g_{ij} as the metric *coefficients* or *components*, as opposed to the metric *tensor*. At the same time, is also often said that the Jacobians are not tensors. However for our purposes the Jacobian coefficients have the same status as the metric coefficients—they are components of the identity tensor, but in a representation that uses a basis and a reciprocal basis drawn from two different coordinate system, as in (30). The equal status of these two sets of coefficients is also apparent if one observes that $g_{ij} \equiv \boldsymbol{e}_i \cdot \boldsymbol{e}_j$ while similarly $J^i_a = \boldsymbol{e}_a \cdot \boldsymbol{e}^i$.

b. Spatial differentiation

Taking the partial spatial derivative of a vector $\boldsymbol{w}$ with respect to the ith coordinate, we find

$$\partial_i \boldsymbol{w} = \partial_i (w^j \boldsymbol{e}_j) = \boldsymbol{e}_j \partial_i w^j + w^j \partial_i \boldsymbol{e}_j \,, \tag{31}$$

which illustrates that the components of the derivative of a tensor are not the derivatives of its components; spatial variations in the basis vectors will contribute via the second term, which itself is equivalent to $w^j \Gamma^k{}_{ij} \boldsymbol{e}_k$, by equation (4). The partial derivative $\partial_i \boldsymbol{w}$ can be rewritten as

$$\partial_i \boldsymbol{w} = \boldsymbol{e}_j \left(\partial_i w^j + \Gamma^j{}_{ik} w^k \right) \equiv \boldsymbol{e}_j \nabla_i w^j \tag{32}$$

Here $\nabla_i w^j$ is known as the *covariant derivative* of w^j, and has been constructed to express the action of taking the partial spatial derivative of a vector in terms of an operation on its components.

The covariant derivative of the components of a tensor takes on a different form depending upon the tensor's order and the valence of its representation. For a scalar or zero-rank tensor field we have the simple relation $\nabla_i a \equiv \partial_i a$, while for a second-order tensor $\boldsymbol{A}$ expressed in mixed component form the covariant derivative is

$$\nabla_i A^j_k \equiv \partial_i A^j_k + \Gamma^j{}_{i\ell} A^\ell_k - \Gamma^\ell{}_{ik} A^j_\ell. \tag{33}$$

This pattern extends in the obvious way to tensor components of different orders and valences, with a positive Christoffel term for each upper index and a negative term for each lower index. Importantly, the covariant derivative exhibits the usual properties, such as the product rule,

$$\nabla_i \left(a^j b_k \right) = a^j \nabla_i b_k + b_k \nabla_i a^j, \tag{34}$$

and linearity; see section 8.6 of Grinfeld (2013).

When it acts on the components of an nth order tensor, the covariant derivative produces components of an $(n+1)$th order tensor, and indeed this was the entire point of defining it this way. This means we can also define the *contravariant derivative*

$$\nabla^i \equiv g^{ij} \nabla_j \,. \tag{35}$$

In Table 1 we see how the covariant derivative is used to write various types of spatial derivatives as actions on tensor components. Importantly, when the coordinate system spans the three dimensions of Euclidean space, covariant differentiation of a tensor component corresponds to partial differentiation of the tensor itself, as is evident from fact that $\nabla_i w^j$ is the jth component of $\partial_i \boldsymbol{w}$ from (32); this will no longer be the case when the covariant derivative is applied along a curved manifold, such as in general relativity or along a curved surface embedded in three dimensional Euclidean space, see Chapter 11 of Grinfeld (2013).

The gradient tensor of a vector $\boldsymbol{w}$ is $\boldsymbol{\nabla} \otimes \boldsymbol{w}$ and has components $\nabla_i w^j$. A notable example is the gradient of the position vector $\boldsymbol{r}$, for which one finds

$$\boldsymbol{\nabla} \otimes \boldsymbol{r} = \boldsymbol{I}, \qquad \nabla_j r^i = \delta^i_j \,, \tag{36}$$

which is readily shown in Cartesian coordinates.[2]

The covariant derivative has the crucial property that its application to the metric coefficients vanishes,

$$\nabla_i g^{jk} = 0, \qquad \nabla_i g_{jk} = 0, \qquad \nabla_i \delta^j_k = 0, \tag{37}$$

a result known as *Ricci's theorem*, *metric compatibility*, or the *metrinilic property*. This means that we can freely raise and lower indices across the derivative. As confirmed in Appendix A, its application to the Jacobian coefficients also vanishes

$$\nabla_i J^a_j = 0, \quad \nabla_i J^j_a = 0, \quad \nabla_a J^i_b = 0, \quad \nabla_a J^b_i = 0, \tag{38}$$

but this had to be the case since it was designed to produce tensor components from tensor components—we have to be able to change bases through the derivative for this to be true.

It may similarly be shown that the application of the covariant derivative to the Levi-Civita tensor vanishes,

$$\nabla_\ell \varepsilon_{ijk} = 0, \qquad \nabla_\ell \varepsilon^{ijk} = 0, \tag{39}$$

meaning that the covariant derivative and the Levi-Civita tensor also commute.

c. Index swapping rules

In changing representations between the $\boldsymbol{e}_i$ and the $\boldsymbol{e}_a$ bases, the following rules will prove useful. Any repeated indices that generate proper tensors may be freely swapped between the $\{i, j, k\}$ and $\{a, b, c\}$ index sets, e.g.,

$$w^i \boldsymbol{e}_i = w^a \boldsymbol{e}_a, \qquad A_i{}^j \boldsymbol{e}^i \otimes \boldsymbol{e}_j = A_i{}^b \boldsymbol{e}^i \otimes \boldsymbol{e}_b \,. \tag{40}$$

[2] In Cartesian coordinates $r^i = \{x, y, z\}$ and $\nabla_j = \partial_j$, so $\nabla_j r^i = \partial_j r^i = \delta^i_j$. It then follows from the tensor transformation law that this holds for any other coordinates y^a: $J^a_i J^j_b \delta^i_j = J^a_i J^i_b = \delta^a_b$.

Any contracted indices on tensor components can likewise be swapped, e.g.,

$$u^i v_i = u^a v_a, \qquad A_{ab} w^b = A_{ai} w^i. \tag{41}$$

For the Christoffel symbols, contractions that involve the upper index and/or one of the two lower indices may be swapped, but not those involving both lower indices:

$$\begin{aligned} \Gamma^i{}_{jk} u_i v^j w^k &= \Gamma^a{}_{jk} u_a v^j w^k = \Gamma^a{}_{bk} u_a v^b w^k \\ &= \Gamma^a{}_{jc} u_a v^j w^c \neq \Gamma^a{}_{bc} u_a v^b w^c. \end{aligned} \tag{42}$$

Although partial derivatives of tensor components are not in general tensor components, indices of *contracted* partial derivatives may also be swapped via

$$u^i \partial_i v^j = u^a \partial_a v^j. \tag{43}$$

This follows from the chain rule and the definitions (25), but we can also use (32) to rewrite expressions involving partial derivatives in terms of the covariant derivative and the Christoffel symbols, for example

$$u^i \partial_i v^j = u^i \nabla_i v^j - \Gamma^j{}_{ik} u^i v^k. \tag{44}$$

We see that swapping the i index for a only requires one of the two lower indices of the Christoffel symbol to change, and is therefore permissible.

4. Time derivatives in curvilinear coordinates

In order to determine the momentum equation in a generally accelerating, rotating, and deforming coordinate system, we need an analogue of the covariant derivative for temporal differentiation. That is, we need a way to compute total time derivatives of tensor fields by operating only on their components. This section introduces such a derivative, first presenting a known result for fixed coordinate systems, and then introducing a generalization that accommodates moving, deforming coordinates.

a. Frames and settings

Consider the coordinate system y^a, which we have introduced as being both rigid and inertial. In fact, both of these properties more properly pertain not to the coordinate system y^a, but rather to the *frame* in which it is embedded. We define a frame to be an imaginary, rigid lattice extending throughout space, in comparison to which Newtonian motion may be measured (see e.g., Koks 2017). Given a non-accelerating frame, any curvilinear coordinate system that is fixed with respect to that frame will inherit its rigidity and inertiality. Thus, referring to y^a as a rigid, inertial *coordinate system* is simply a shorthand for saying that it is a coordinate system embedded within an inertial *frame*.

TABLE 4. A table of different time derivatives appearing in this work. χ is a general path along which differentiation occurs. We understand the material derivative of a tensor to be taken following a fixed parcel label ξ. The intrinsic derivatives are constructed to produce components of the corresponding total derivatives acting on tensors. One further sort of time derivative, the convected derivative of Oldroyd (1950), is discussed in Appendix B and omitted here.

Name	Symbol	Argument
Total along path χ	$\frac{d}{dt}\big\vert^{\chi}$	Tensor
Total Material	$\frac{d}{dt} \equiv \frac{d}{dt}\big\vert^{\chi_\xi} \equiv \frac{d}{dt}\big\vert_{\xi}$	Tensor
Partial with fixed x	$\frac{\partial}{\partial t}\big\vert_{x}$	Either
Partial with fixed y	$\frac{\partial}{\partial t}\big\vert_{y}$	Either
Covariant Advection	$\nabla_{\boldsymbol{v}} \equiv v^k \nabla_k = v^c \nabla_c$	Components
Lie Advection	$\mathcal{L}_{\boldsymbol{\mathcal{V}}}$	Components
Intrinsic	$\frac{\delta}{\delta t}\big\vert^{\chi}$	Components
Intrinsic Material	$\frac{\delta}{\delta t} \equiv \frac{\delta}{\delta t}\big\vert^{\chi_\xi} \equiv \frac{\delta}{\delta t}\big\vert_{\xi}$	Components

Recall that we have employed Galilean invariance to regard this inertial frame as stationary.

The x^i coordinate surfaces, by contrast, are allowed to evolve with respect to the inertial frame according to any combination of translation, rotation, and deformation. Because these coordinates are permitted to exhibit non-rigid motion, it is not possible to fix them within any rigid frame. We instead introduce the term *setting* to indicate a deformable frame-like entity, visualizable in two dimensions as an evolving, stretching rubber sheet onto which coordinate lines are marked; in the case without deformation, a setting becomes a frame. For brevity we refer to the x^i coordinates as being within the *moving setting*.

Since the x^i coordinate surfaces are allowed to be temporally varying, it follows that the basis vectors $\boldsymbol{e}_i$, metric coefficients g_{ij} and g^{ij}, and Christoffel symbols $\Gamma^k{}_{ij}$ vary as well. Meanwhile, the y^a coordinate surfaces and hence basis vectors $\boldsymbol{e}_a$, metric coefficients g_{ab} and g^{ab}, and Christoffel symbols $\Gamma^c{}_{ab}$ are all temporally constant. The Jacobian coefficients J^a_i and J^i_a for converting between the two coordinate systems will also vary in time as the x^i coordinates do. All of these quantities are permitted to be spatially varying.

The two coordinate systems are related as follows, sketched in Fig. 1. The origin $O_x(t)$ of the coordinate system in the moving setting is located at position vector $\boldsymbol{S}(t)$ with respect to the origin O_y of the coordinate system in the inertial frame. Let $\boldsymbol{r}$ be a position vector with respect to the origin of the moving coordinate system O_x, and $\boldsymbol{s}$ a position vector with respect to the origin of the inertial coordinate system O_y, with $\boldsymbol{s} = \boldsymbol{r} + \boldsymbol{S}$.

It is assumed that our coordinate systems are sufficiently well-behaved that they can be related by invertible sets of differentiable functions, allowing one to translate freely between the names given to a particular point in each system. That is, we can write each coordinate system in terms of the other as

$$x^i(y,t), \qquad y^a(x,t), \tag{45}$$

where we suppress coordinate indices within function arguments. Differentiating each set of coordinates with the other held fixed yields two velocity fields, $\boldsymbol{\mathcal{U}}$ and $\boldsymbol{\mathcal{V}}$, having components

$$\mathcal{U}^i \equiv \left.\frac{\partial x^i}{\partial t}\right|_y, \qquad \mathcal{V}^a \equiv \left.\frac{\partial y^a}{\partial t}\right|_x. \tag{46}$$

At each point in space $\boldsymbol{\mathcal{U}} = \mathcal{U}^i \boldsymbol{e}_i$ is the velocity of the inertial frame as observed from the moving setting, while $\boldsymbol{\mathcal{V}} = \mathcal{V}^a \boldsymbol{e}_a$ is the velocity of the moving setting as observed from the inertial frame.

Clearly these velocity fields are equal and opposite, $\boldsymbol{\mathcal{U}} = -\boldsymbol{\mathcal{V}}$, although they both may vary in space and time. Following the tensor transformation law, each field is expressible in terms of the other basis as

$$\mathcal{U}^a \equiv J^a_i \mathcal{U}^i, \qquad \mathcal{V}^i \equiv J^i_a \mathcal{V}^a, \tag{47}$$

where $J^a_i \equiv \partial_i y^a$ and $J^i_a \equiv \partial_a x^i$ are the Jacobian coefficients defined earlier in (25), such that $\boldsymbol{\mathcal{V}} = \mathcal{V}^i \boldsymbol{e}_i = \mathcal{V}^a \boldsymbol{e}_a$, and similarly for $\boldsymbol{\mathcal{U}}$. Because the inertial coordinate system is assumed to be stationary, $\boldsymbol{\mathcal{V}}$ is the absolute velocity of the moving setting. Hereafter we use only $\boldsymbol{\mathcal{V}}$, which we name the *setting velocity field*[3].

Let the scalar field ξ be a label that remains constant following fluid parcels. Differentiating each set of coordinates with the label ξ held fixed yields two velocity fields characterizing the fluid motion, $\boldsymbol{u} = u^i \boldsymbol{e}_i$ and $\boldsymbol{v} = v^a \boldsymbol{e}_a$, with components given by

$$u^i \equiv \left.\frac{dx^i}{dt}\right|_\xi, \qquad v^a \equiv \left.\frac{dy^a}{dt}\right|_\xi. \tag{48}$$

The velocity field $\boldsymbol{u}$ is the *relative velocity* observed with respect to the moving setting while $\boldsymbol{v}$ is the *absolute velocity*. Although these velocity fields make the most sense

[3] A particular example of a setting velocity can be found in Griffies et al. (2020) as $\boldsymbol{v}^{(s)}$, described as "the velocity of a point on the s-coordinate surface." As in equation (51), it is subtracted from the absolute fluid velocity to calculate the relative dia-surface velocity $w^{(s)}$, corresponding here to the vertical component of the relative velocity $\boldsymbol{u}$.

in the setting where they are easily observed, they are vector fields and can therefore be expressed in components relative to the basis vectors of the other setting,

$$u^a \equiv J^a_i u^i, \qquad v^i \equiv J^i_a v^a. \tag{49}$$

These two velocity fields are related as follows: considering the x^i coordinates to be a function of parcel label and time, $x^i(\xi,t)$, we have

$$\left.\frac{d}{dt} y^a[x(\xi,t),t]\right|_\xi = J^a_i \left.\frac{dx^i}{dt}\right|_\xi + \left.\frac{\partial y^a}{\partial t}\right|_x, \tag{50}$$

which reduces to $v^a = u^a + \mathcal{V}^a$. Consequently

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{\mathcal{V}}. \tag{51}$$

The absolute velocity of the fluid $\boldsymbol{v}$ is thus the sum of its velocity $\boldsymbol{u}$ relative to the moving setting and the setting velocity $\boldsymbol{\mathcal{V}}$ due to the motion of the x setting itself. This is precisely what we should expect.

If the coordinate system x^i is embedded within a frame of reference rotating with the planet, then $\boldsymbol{\mathcal{V}}$ is the planetary velocity and $\boldsymbol{u}$ is the velocity relative to the planet. In cases where the x^i coordinates deform relative to the planet, for example in isopycnal or pressure coordinates, the setting velocity $\boldsymbol{\mathcal{V}}$ captures this coordinate motion as well as the planetary motion.

b. The intrinsic derivative in static frames

Next we present a standard indicial expression for the derivative along an arbitrary curve, the *intrinsic derivative*, which allows us to express the material derivative following fluid parcels as a special case. The intrinsic derivative is defined in the literature only for *static* coordinate systems and requires generalization to accommodate rotating or deforming coordinate systems x^i.

Let $\chi(t)$ be some parametrized curve, with coordinates $\chi^a(t)$ and $\chi^i(t)$ within the y^a and x^i coordinate systems respectively, and write $f[\chi(t),t]$ to denote the restriction of the field f to the curve χ. We use the notation

$$\left.\frac{df}{dt}\right|^\chi \equiv \frac{d}{dt} f[\chi(t),t] \tag{52}$$

to mean the total time derivative of f evaluated along χ, which is independent of the coordinates we use to represent χ. This notation is intended to be distinct from $df/dt|_Q$, which means that the derivative is taken with the quantity Q held at a fixed value.

The rates of change of the x^i and y^a coordinates as one moves along the curve χ specify two sets of tangent vectors, $\boldsymbol{p}[\chi(t)] = p^i \boldsymbol{e}_i$ and $\boldsymbol{q}[\chi(t)] = q^a \boldsymbol{e}_a$, called the *parametric velocities*, having components

$$p^i \equiv \left.\frac{dx^i}{dt}\right|^\chi, \qquad q^a \equiv \left.\frac{dy^a}{dt}\right|^\chi. \tag{53}$$

These give the apparent velocity of motion following the parameterized curve as observed with respect to the moving setting and the inertial frame, respectively. We emphasize that these are *not* material velocities—they correspond to the rate of change of a position along any chosen curve $\chi(t)$, and are in general completely independent of the velocity of the fluid in which the curve is embedded. By reasoning analogous to (50) and (51), we arrive at

$$\boldsymbol{q} = \boldsymbol{p} + \boldsymbol{\mathcal{V}}. \tag{54}$$

When f is considered to be a function of the y^a coordinates, $f = f[\chi^a(t),t]$, its derivative along χ is

$$\left.\frac{df}{dt}\right|^\chi = \left.\frac{\partial f}{\partial t}\right|_y + q^b \partial_b f \tag{55}$$

from the chain rule. When applied to a vector field $\boldsymbol{w}$, the total derivative along χ can be expressed as

$$\begin{aligned}\left.\frac{d\boldsymbol{w}}{dt}\right|^\chi &= \left.\frac{\partial \boldsymbol{w}}{\partial t}\right|_y + q^b \partial_b \boldsymbol{w} \\ &= \boldsymbol{e}_a \left(\left.\frac{\partial}{\partial t}\right|_y + q^b \nabla_b \right) w^a\end{aligned} \tag{56}$$

from the fact that the basis vectors $\boldsymbol{e}_a$ are temporally constant combined with the definition of the covariant derivative in (32). We can thus define the *intrinsic derivative*

$$\left.\frac{\delta}{\delta t}\right|^\chi \equiv \left(\left.\frac{\partial}{\partial t}\right|_y + q^b \nabla_b \right), \tag{57}$$

acting on tensor components, allowing us to write

$$\left.\frac{d\boldsymbol{w}}{dt}\right|^\chi = \left.\frac{\delta w^a}{\delta t}\right|^\chi \boldsymbol{e}_a. \tag{58}$$

See §7.55 of Aris (1962) or §8.9 of Grinfeld (2013) for extended discussion of this operator.

The action of the intrinsic derivative is to express the total time derivative along the curve $\chi(t)$ in static but curvilinear coordinate systems in terms of operations on tensor components alone. Put another way: when we apply the intrinsic derivative to the components of a tensor, what we are doing is taking the total time derivative of that tensor along the specified curve. The form of the intrinsic derivative given in (57) is independent of the order and valence of the tensor components on which it operates because this dependence is already accounted for by the covariant derivative. Finally, we reiterate that equation

(57) is valid only for tensor components with respect to fixed basis vectors.

c. The material derivative in static frames

We now choose a curve, denoted $\chi_\xi(t)$, to follow the fluid parcel ξ, and then allow ξ to take on all possible label values, such that $\chi_\xi(t)$ specifies the paths of all fluid parcels. The total derivative along the curve $\chi_\xi(t)$ following fluid parcels for each value of ξ—that is, the *material derivative*—of some tensor quantity f is then given by

$$\left.\frac{df}{dt}\right|^{\chi_\xi} = \frac{d}{dt} f[\chi_\xi(t), t]. \tag{59}$$

From (53), the parametric velocity components p^i and q^a evaluated along the paths of fluid parcels become

$$p^i|^{\chi_\xi} \equiv \left.\frac{dx^i}{dt}\right|_\xi = u^i, \qquad q^a|^{\chi_\xi} \equiv \left.\frac{dy^a}{dt}\right|_\xi = v^a. \tag{60}$$

from (48). This emphasizes that the fluid velocities $\boldsymbol{u}$ and $\boldsymbol{v}$ are a special case of the parametric velocities $\boldsymbol{p}$ and $\boldsymbol{q}$ along a family of curves, when those curves are taken to follow fluid parcels.

We define the *material intrinsic derivative*

$$\frac{\delta}{\delta t} \equiv \left(\left.\frac{\partial}{\partial t}\right|_y + v^b \nabla_b \right) \tag{61}$$

Applied to a vector field $\boldsymbol{w}$, this is written in terms of the inertial y^a coordinates as

$$\left.\frac{d\boldsymbol{w}}{dt}\right|^{\chi_\xi} = \frac{\delta w^a}{\delta t} \boldsymbol{e}_a, \tag{62}$$

This is simply the intrinsic derivative (57) evaluated along fluid parcel trajectories, for which the parametric velocity $\boldsymbol{q}$ has been set to the absolute fluid velocity $\boldsymbol{v}$, and again the form (61) is only valid for static coordinate systems. For convenience, we define the undecorated symbol $\delta/\delta t$ to be the particular, material case of the intrinsic derivative along a curve because this version is most common in our applications. If we need to differentiate along some other path χ, we denote that explicitly by "$(\cdot)|^\chi$."

d. Velocities and settings

We pause momentarily to investigate the relationship between the vector fields $\boldsymbol{p}$ and $\boldsymbol{q}$, and to clarify the notion of a setting. As in Fig. 1, let $\boldsymbol{r}$ and $\boldsymbol{s}$ be the position vectors measured from the origins O_x and O_y respectively to some point in space. Due to the motion of the basis vectors, $\boldsymbol{r}$ may change even with coordinates x^i held fixed. However, on account of our choice that the inertial coordinate system y^a be embedded within a stationary frame, $\boldsymbol{s}$ may *not* change with y^a held fixed. Consequently, considering the position vectors along the curve $\chi(t)$, we see that $\boldsymbol{r}[\chi(t), t]$ is an explicit function of time but $\boldsymbol{s}[\chi(t)]$ is not.

Taking the time derivatives of these position vector along the curve $\chi(t)$ then leads to

$$\left.\frac{d\boldsymbol{r}}{dt}\right|^\chi = \boldsymbol{p} + \left.\frac{\partial \boldsymbol{r}}{\partial t}\right|_x, \qquad \left.\frac{d\boldsymbol{s}}{dt}\right|^\chi = \boldsymbol{q} \tag{63}$$

from $\boldsymbol{e}_i \equiv \partial_i \boldsymbol{r}$ and $\boldsymbol{e}_a \equiv \partial_a \boldsymbol{s}$, together with the definitions of $\boldsymbol{p}$ and $\boldsymbol{q}$ in (53). Thus the rate of change of the position vector $\boldsymbol{s}$ measured with respect to the inertial frame (which has been chosen to be stationary) is the same as the vector field $\boldsymbol{q}$ observed in that frame, but the same is not true for the moving setting. In the moving setting, an additional contribution $\partial \boldsymbol{r}/\partial t|_x$ appears due to the motion of the coordinate surfaces themselves. We name $\partial \boldsymbol{r}/\partial t|_x$ the *inherent velocity field* since it is an inherent property of the setting, not of the fluid motion. The simplest case of a non-zero inherent velocity field is that due to rigid rotation.

We may now make the following key observation with respect to settings: if two different coordinate systems have the same inherent velocity fields, they share the same setting. In this case, all coordinate locations in one coordinate system remain at the same coordinate locations in the other system even as the setting moves or deforms. These two coordinate systems can then be visualized in two dimensions as two different sets of lines on the same deformable rubber sheet. Like a frame, a setting therefore does not depend upon or imply a choice of coordinates, even if a particular coordinate system was involved in its initial specification.

The inherent velocity field $\partial \boldsymbol{r}/\partial t|_x$ is related to the setting velocity $\boldsymbol{\mathcal{V}}$, as will now be shown. Since $\boldsymbol{s} = \boldsymbol{r} + \boldsymbol{S}$, see Fig. 1, we can take the total derivative of this equation along curves $\chi(t)$ to yield, using (63) and (60),

$$\boldsymbol{q} = \boldsymbol{p} + \left.\frac{\partial \boldsymbol{r}}{\partial t}\right|_x + \frac{d\boldsymbol{S}}{dt}. \tag{64}$$

Since $\boldsymbol{S}$ is spatially constant for a given time, we also have

$$\frac{d\boldsymbol{S}}{dt} = \left.\frac{\partial \boldsymbol{S}}{\partial t}\right|_x = \left.\frac{\partial \boldsymbol{S}}{\partial t}\right|_y \tag{65}$$

Comparison with (54) then shows

$$\boldsymbol{\mathcal{V}} = \left.\frac{\partial \boldsymbol{r}}{\partial t}\right|_x + \frac{d\boldsymbol{S}}{dt} = \left.\frac{\partial \boldsymbol{s}}{\partial t}\right|_x, \tag{66}$$

which means that the setting velocity field $\boldsymbol{\mathcal{V}}$ consists of the inherent velocity of the moving setting $\partial \boldsymbol{r}/\partial t|_x$ plus $d\boldsymbol{S}/dt$, the velocity of the origin of the moving setting. This is equivalent to the apparent velocity of the absolute location vector $\boldsymbol{s}$ as viewed from the moving setting. In the common case that x^i and y^a have coincident origins, this reduces to $\boldsymbol{\mathcal{V}} = \partial \boldsymbol{r}/\partial t|_x = \partial \boldsymbol{s}/\partial t|_x$.

e. Transforming rates of change

In order to derive the intrinsic derivative in the moving setting we must be able to transform partial time derivatives between inertial frames and moving settings. The partial time derivative of a function f at a fixed x location is, by the chain rule,

$$\left.\frac{\partial f}{\partial t}\right|_x = \left.\frac{\partial f}{\partial t}\right|_y + \mathcal{V}^a \partial_a f \tag{67}$$

where f is regarded first as a function of the x^i and then of the y^a coordinates. Thus we have

$$\left.\frac{\partial}{\partial t}\right|_x = \left(\left.\frac{\partial}{\partial t}\right|_y + \mathcal{V}^b \partial_b\right) \tag{68}$$

where the operand field on the left-hand side is understood to depend explicitly on x while that on the right-hand side is understood to depend explicitly on y, as indicated respectively by the "$|_x$" and "$|_y$" notation. Proceeding as above but for the derivative $\partial f/\partial t|_y$, one finds

$$\left.\frac{\partial}{\partial t}\right|_y = \left(\left.\frac{\partial}{\partial t}\right|_x - \mathcal{V}^j \partial_j\right) \tag{69}$$

as the inverse transformation to (68). These relationships are a consequence only of functional dependence and therefore hold for general functions, which might be tensors of any order, their components, or even non-tensor functions like the Christoffel symbols.

While the rates of change of the basis vectors $\boldsymbol{e}_a$ at fixed y vanish, those of the basis vectors at fixed x are found to be

$$\left.\frac{\partial \boldsymbol{e}_i}{\partial t}\right|_x = \boldsymbol{e}_j \nabla_i \mathcal{V}^j \qquad \text{and} \qquad \left.\frac{\partial \boldsymbol{e}^i}{\partial t}\right|_x = -\boldsymbol{e}^j \nabla_j \mathcal{V}^i. \tag{70}$$

Temporal changes in the bases and spatial gradients of $\boldsymbol{\mathcal{V}}$ are then just two representations of the same phenomenon.[4] We may also note the useful fact that the temporal derivative of the setting velocity $\boldsymbol{\mathcal{V}}$ at a fixed location in the inertial frame can be expressed as

$$\boldsymbol{e}^a \cdot \left.\frac{\partial \boldsymbol{\mathcal{V}}}{\partial t}\right|_y = \left.\frac{\partial \mathcal{V}^a}{\partial t}\right|_y = J^a_i \left.\frac{\partial \mathcal{V}^i}{\partial t}\right|_x. \tag{72}$$

Thus these follow the tensor transformation law for vector components. Notice that the coordinates held fixed during differentiation differ on either side of the second equality. This property is particular to the setting velocity $\boldsymbol{\mathcal{V}}$ and is not true for tensors in general.[5] This point will be crucial when we come to the momentum equation (102).

f. The intrinsic derivative in moving settings

We are now able to generalize the intrinsic derivative to a rotating, deforming setting. The derivative of vector field $\boldsymbol{w}$ along a curve χ, as expressed in terms of its contravariant components in the moving coordinate system x^i, is

$$\left.\frac{d\boldsymbol{w}}{dt}\right|^\chi = \left(\left.\frac{\partial}{\partial t}\right|_x + p^j \partial_j\right)\left(w^i \boldsymbol{e}_i\right) = \left.\frac{\partial w^i}{\partial t}\right|_x \boldsymbol{e}_i + p^j \partial_j \boldsymbol{w} + w^i \left.\frac{\partial \boldsymbol{e}_i}{\partial t}\right|_x \tag{75}$$

using the definition of p^i in (53). The final term is of a new sort wholly absent in static frames, though common in GFD settings, and gives the contribution to the rate of change of $\boldsymbol{w}$ along $\chi(t)$ arising from the temporal variability of the basis vectors.

The intrinsic derivative of the contravariant components of a vector in a moving coordinate system is then defined as

$$\left.\frac{\delta w^i}{\delta t}\right|^\chi \equiv \left.\frac{\partial w^i}{\partial t}\right|_x + p^j \nabla_j w^i + w^j \nabla_j \mathcal{V}^i \tag{76}$$

which supersedes the earlier definition in (57). The new, final term emerges from the rate of change of the basis vectors in (75) on account of (70), and vanishes for any static coordinate system, recovering the form of the intrinsic derivative given earlier in (57). With this definition, we

[4] As a brief proof, we note

$$\left.\frac{\partial \boldsymbol{e}_i}{\partial t}\right|_x = \left.\frac{\partial}{\partial t}\right|_x \frac{\partial \boldsymbol{r}}{\partial x^i} = \frac{\partial}{\partial x^i}\left.\frac{\partial \boldsymbol{r}}{\partial t}\right|_x$$

just by reversing the order of the partial derivatives. Substituting from (66), this becomes

$$\left.\frac{\partial \boldsymbol{e}_i}{\partial t}\right|_x = \partial_i\left(\boldsymbol{\mathcal{V}} - \frac{d\boldsymbol{S}}{dt}\right) = \partial_i \boldsymbol{\mathcal{V}} = \boldsymbol{e}_j \nabla_i \mathcal{V}^j$$

using the fact that $\boldsymbol{S}$ is spatially constant together with the definition of the covariant derivative in (32). The expression for the contravariant basis follows from this and the fact that

$$\left.\frac{\partial \delta^i_j}{\partial t}\right|_x = 0. \tag{71}$$

[5] The first equality in (72) is true because the $\boldsymbol{e}_a$ are constant in time, and to verify the second equality we use (69) to write

$$\left.\frac{\partial \boldsymbol{\mathcal{V}}}{\partial t}\right|_y = \left.\frac{\partial}{\partial t}\right|_x \left(\mathcal{V}^i \boldsymbol{e}_i\right) - \mathcal{V}^i \partial_i \boldsymbol{\mathcal{V}} = \left.\frac{\partial \mathcal{V}^i}{\partial t}\right|_x \boldsymbol{e}_i + \mathcal{V}^i \left(\left.\frac{\partial \boldsymbol{e}_i}{\partial t}\right|_x - \boldsymbol{e}_j \nabla_i \mathcal{V}^j\right) \tag{73}$$

after using (32) in passing to the second equality. The term in parentheses then vanishes due to (70), and dotting both sides with $\boldsymbol{e}_a$ yields (72). Separately, we can use equation (68) and Table A1 to write, for a general vector $\boldsymbol{w}$,

$$J^a_i \left.\frac{\partial w^i}{\partial t}\right|_x = \left(\left.\frac{\partial}{\partial t}\right|_y + \mathcal{L}_{\boldsymbol{\mathcal{V}}}\right) w^a \tag{74}$$

from which equation (72) follows immediately by $\mathcal{L}_{\boldsymbol{\mathcal{V}}} \mathcal{V}^a = 0$. In fact, with the choice $\boldsymbol{\mathcal{V}} = \boldsymbol{v}$, equation (74) becomes Oldroyd (1950) equation (23) applied to vector components—see Appendix B for further discussion of that derivative and its relation to the intrinsic derivative.

can express the total derivative along $\chi(t)$ in the moving coordinate system as

$$\left.\frac{d\boldsymbol{w}}{dt}\right|^{\mathcal{X}} = \left.\frac{\delta w^i}{\delta t}\right|^{\mathcal{X}} \boldsymbol{e}_i \tag{77}$$

just as was the case for the static coordinate system in (58).

Unlike for the static coordinate system, the expression for the intrinsic derivative in a moving coordinate system is valence-dependent. By an analogous procedure, we find the derivative for covariant components of a vector to be

$$\left.\frac{\delta w_i}{\delta t}\right|^{\mathcal{X}} \equiv \left.\frac{\partial w_i}{\partial t}\right|_x + p^j \nabla_j w_i - w_j \nabla_i \mathcal{V}^j, \tag{78}$$

In comparison with (76) for the contravariant components, we see that the final term appears with the opposite sign and with a different contraction of indices. Equations (76) and (78) indeed give the contravariant and covariant components of the same vector $d\boldsymbol{w}/dt|^{\mathcal{X}}$, which is most easily shown using the general form of the derivative presented in section 4h. Despite this, neither the first nor third terms on their right-hand sides are individually the components of the same vector. The implications of this are discussed in section 5g.

For a second-order tensor $\boldsymbol{A}$ expressed in contravariant form, $\boldsymbol{A} = A^{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$, one finds the intrinsic derivative in a moving coordinate system to be

$$\left.\frac{\delta A^{ij}}{\delta t}\right|^{\mathcal{X}} = \left.\frac{\partial A^{ij}}{\partial t}\right|_x + p^k \nabla_k A^{ij} + A^{kj}\nabla_k \mathcal{V}^i + A^{ik}\nabla_k \mathcal{V}^j. \tag{79}$$

This pattern is expressed most naturally in terms of the Lie derivative along $\boldsymbol{\mathcal{V}}$, which captures all of the valence dependence in its definition. Find details in section 4h.

Like the covariant spatial derivative, the intrinsic derivative obeys the tensor transformation law

$$\left.\frac{\delta w^i}{\delta t}\right|^{\mathcal{X}} = J^i_a \left.\frac{\delta w^a}{\delta t}\right|^{\mathcal{X}} \tag{80}$$

Separately, it is compatible with the metric, and so indices can be raised and lowered across the derivative like so

$$\left.\frac{\delta w_i}{\delta t}\right|^{\mathcal{X}} = g_{ij} \left.\frac{\delta w^j}{\delta t}\right|^{\mathcal{X}} \tag{81}$$

We prove this in section 4h, and it is standard for the derivative in fixed frames.

g. The material derivative in moving settings

The *material intrinsic derivative* following fluid parcels is then

$$\frac{\delta w^i}{\delta t} \equiv \left.\frac{\partial w^i}{\partial t}\right|_x + u^j \nabla_j w^i + w^j \nabla_j \mathcal{V}^i, \tag{82}$$

$$\frac{\delta w_i}{\delta t} \equiv \left.\frac{\partial w_i}{\partial t}\right|_x + u^j \nabla_j w_i - w_j \nabla_i \mathcal{V}^j, \tag{83}$$

found by choosing a family of curves χ_ξ as parcel trajectories relative to the coordinates such that $u^i = p^i|_\xi$ as in subsection 4c, but retaining the final terms due to the moving basis. An equation equivalent to (82) appears as equation (23) of Luo and Bewley (2004), but the expression is not further generalized.

The first two terms on the right-hand side of (82) give the apparent rate of change of $\boldsymbol{w}$ following the flow $\boldsymbol{u}$ as observed from within the moving setting, measuring the contravariant components of $\boldsymbol{w}$. The final term tells us that at a fixed x location, the changing basis vectors interact with the vector field $\boldsymbol{w}$ to act *as if* the setting velocity field $\boldsymbol{\mathcal{V}}$ were being advected by $\boldsymbol{w}$. We term this effect *counter-advection*.

An illuminating special case of counter-advection is that in which the vector field $\boldsymbol{w}$ is both spatially and temporally constant. In that case, temporal changes in the contravariant components w^i at fixed x locations must occur to counteract the fact that the basis vectors are also changing. The combined effect leaves the derivative of the $\boldsymbol{w}$ field unchanged.

h. The intrinsic derivative in its most general form

Ultimately, we want a form of this derivative that abstracts away its dependence on the order and valence of the tensor being differentiated. To achieve this, we need an operation that undoes the apparent effect of the moving coordinates being dragged around by the field $\boldsymbol{\mathcal{V}}$. Such an operation already exists: the Lie derivative, which tells us how a tensor field changes as it is "pulled back" by the flow along the integral curves of a vector field, in this case the setting velocity $\boldsymbol{\mathcal{V}}$. The Lie derivative is a potent tool in differential geometry, and appears as a generalized advection operator in the work of Holm (e.g., 2015, 2025). The intrinsic derivative, it turns out, also takes its most powerful form in terms of the Lie derivative:

$$\left.\frac{\delta}{\delta t}\right|^{\mathcal{X}} = \left(\left.\frac{\partial}{\partial t}\right|_x + q^j \nabla_j - \mathcal{L}_{\boldsymbol{\mathcal{V}}}\right), \tag{84}$$

a form which is now independent of the valence of its argument. For the contravariant components of a vector, covariant components of vector, and contravariant components of a second-order tensor, the Lie derivative is given respectively by

$$\mathcal{L}_{\boldsymbol{\mathcal{V}}} w^i = \mathcal{V}^j \partial_j w^i - w^j \partial_j \mathcal{V}^i, \tag{85}$$

$$\mathcal{L}_{\boldsymbol{\mathcal{V}}} w_i = \mathcal{V}^j \partial_j w_i + w_j \partial_i \mathcal{V}^j, \tag{86}$$

$$\mathcal{L}_{\boldsymbol{\mathcal{V}}} A^{ij} = \mathcal{V}^k \partial_k A^{ij} - A^{kj}\partial_k \mathcal{V}^i - A^{ik}\partial_k \mathcal{V}^j. \tag{87}$$

The general form for the Lie derivative may be found in (B.18) of Carroll (2004), or likely in any text on differen-

tial geometry or general relativity. Due to the symmetry of the Christoffel symbols, the partial derivatives can equivalently be written as covariant derivatives. The reader may verify that equation (84) recovers all of the particular cases (76), (78), and (79). In the inertial case x becomes y, we take $\boldsymbol{\mathcal{V}} = 0$, and recover equation (57) as well.

As always, the choice $\boldsymbol{q} = \boldsymbol{v}$ gives the material version

$$\frac{\delta}{\delta t} = \left(\left.\frac{\partial}{\partial t}\right|_x + v^j \nabla_j - \mathcal{L}_{\boldsymbol{\mathcal{V}}} \right) \tag{88}$$

We might have guessed this result from the start: as implied by equations (45), the criterion for a well-behaved moving coordinate system x^i is that the setting velocity $\boldsymbol{\mathcal{V}}$ be a vector field that generates a diffeomorphism—that is, an invertible, one-to-one mapping—from space onto itself, as illustrated in Figure 2. By definition, the Lie derivative $\mathcal{L}_{\boldsymbol{\mathcal{V}}}$ is the derivative that measures how a tensor field changes under the diffeomorphism generated by $\boldsymbol{\mathcal{V}}$, and by subtracting it we undo the artifacts that this coordinate transformation introduces to tensor components.

Said another way, the observed velocity $\boldsymbol{u}$ is a transformation of space—a rearrangement of the fluid—that we can meaningfully decompose into its active part $\boldsymbol{v}$, the field along which the fluid moves, and its passive part $\boldsymbol{\mathcal{V}}$, the field along which the coordinates move through the fluid. Instead of simple advection $(\boldsymbol{u} \cdot \boldsymbol{\nabla})$ by the observed velocity $\boldsymbol{u}$, each part of $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{\mathcal{V}}$ effects a different sort of advection: active, covariant advection by the inertial fluid velocity $\boldsymbol{v}$ and passive, Lie advection opposite the setting velocity $\boldsymbol{\mathcal{V}}$, which is the source of the apparent forces.

Geometrically meaningful expressions now tumble from the notation. The expression

$$\left.\frac{\partial}{\partial t}\right|_x - \mathcal{L}_{\boldsymbol{\mathcal{V}}} \tag{89}$$

measures the difference between the observed change with time at a fixed x coordinate location and the change due only to the rearrangement of coordinates induced by $\boldsymbol{\mathcal{V}}$. This operation is metric compatible because the actions of $\partial/\partial t|_x$ and $\mathcal{L}_{\boldsymbol{\mathcal{V}}}$ on the metric coefficients give the same value

$$\left.\frac{\partial g_{ij}}{\partial t}\right|_x = \mathcal{L}_{\boldsymbol{\mathcal{V}}} g_{ij} = 2\nabla_{(i}\mathcal{V}_{j)} \tag{90}$$

and hence (89) applied to g_{ij} vanishes. Together with the metric compatibility of the covariant derivative this proves metric compatibility of the generalized intrinsic derivative from (84),

$$\left.\frac{\delta g_{ij}}{\delta t}\right|^{\mathcal{X}} = 0. \tag{91}$$

This property is important because it allows us to freely raise and lower indices, finally proving that

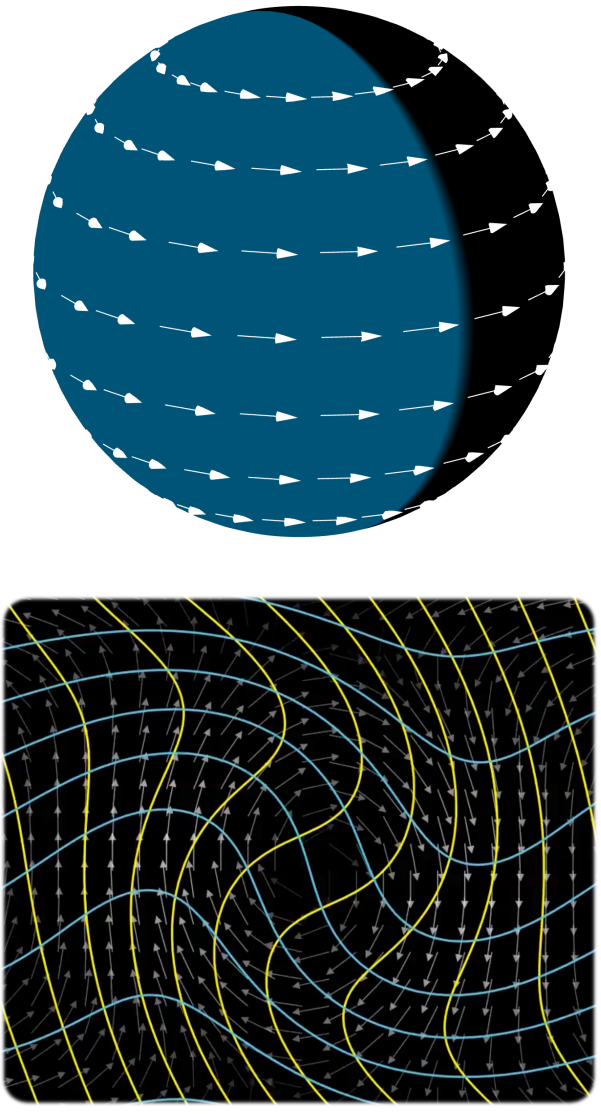


FIG. 2. The setting velocity $\boldsymbol{\mathcal{V}}$, shown in white, is a vector field that generates a diffeomorphism of space. **Top**: $\boldsymbol{\mathcal{V}} = \boldsymbol{\Omega} \times \boldsymbol{r}$ generates rotation of the sphere. In this case, $\boldsymbol{\mathcal{V}}$ is a Killing vector field because rotation is a rigid transformation for which $\nabla_{(i}\mathcal{V}_{j)} = 0$. **Bottom**: A swirling and shearing diffeomorphism of the plane. Here $\nabla_{(i}\mathcal{V}_{j)} \neq 0$, and we see a straining deformation. There is no divergent deformation because this $\boldsymbol{\mathcal{V}}$ was generated from a stream function, so $\nabla_i \mathcal{V}^i = 0$.

$$\left.\frac{\delta w_i}{\delta t}\right|^{\mathcal{X}} = g_{ij} \left.\frac{\delta w^j}{\delta t}\right|^{\mathcal{X}}. \tag{81}$$

Metric compatibility of $\delta/\delta t|^{\mathcal{X}}$ is expected because we constructed it to give components of the vector field $d\boldsymbol{w}/dt$ when acting either on w^i or w_i. Nevertheless, it is reassuring to demonstrate this property directly. See Table A1 for a compilation of derivatives of metric coefficients.

Equation (90) also means that if $\nabla_{(i}\mathcal{V}_{j)}$ vanishes—if the gradient tensor of $\mathcal{V}$ is totally antisymmetric, which is a defining characteristic of a rigid frame (108)—then the derivatives in (89) are *separately* metric compatible, $\mathcal{V}$ satisfies *Killing's equation*

$$\mathcal{L}_{\mathcal{V}} g_{ij} = 0, \tag{92}$$

and the fantastical force vanishes. The setting velocity $\mathcal{V}$ then generates a rigid transformation of space if and only if it is a Killing vector field, motivating our choice in equation (108) to denote a rigid setting velocity by $\mathcal{K}$. A Killing field is a vector field representing an isometry (a distance-preserving transformation), and in Lorentzian spacetime correspond to quantities conserved along geodesics. See §3.8 and Appendix B of Carroll (2004) for a discussion of Killing vectors and symmetries of the metric tensor.

5. Applications of the intrinsic derivative to the momentum equation

The coordinate systems employed in geophysical fluid dynamics, such as isopycnal coordinates, are often changing in time. In this section we develop the equations for conservation of momentum in arbitrary time-dependent coordinates using the intrinsic material derivative from the previous section, and present interpretations of the novel terms which arise. These equations are directly relevant to ocean modeling.

a. The apparent acceleration

We can think of observing velocity as the process of marking the coordinates of successive positions over a small time interval and taking their difference. In two dimensions the coordinates can be visualized as lines drawn on a deformable rubber sheet. If the coordinate lines are moving, this is unknown to an observer who has only these lines themselves as a reference. Similarly, acceleration is observed by a second difference of these position marks, and again, contributions to the acceleration that arise from the motion of the coordinates are invisible to an observer in the moving setting. Separately, acceleration may be measured through an accelerometer, or felt viscerally in the body. When in a moving setting, the measured, or felt accelerations that would not occur in a Newtonian rigid frame constitute the apparent forces, precisely the terms in (102) containing $\mathcal{V}$.

The two terms on the left-hand side of (102) describe the acceleration observed from the moving setting as one moves with the relative velocity $\boldsymbol{u}$, and records contravariant components of velocity. The first, $\partial u^i/\partial t\big|_x$, is local rate of change of the contravariant components of velocity relative to the moving setting observed at a fixed coordinate location in that setting. The second, $u^j\nabla_j u^i$, is ith component of the advective term $\boldsymbol{e}^i\cdot(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u}$, the rate of change of relative velocity $\boldsymbol{u}$ that arises from moving through gradients of the relative velocity at the relative velocity itself. Contributions to the absolute velocity $\boldsymbol{v}$ that involve the setting velocity $\mathcal{V}$ will not be apparent to an observer who quantifies motion based only on references to the setting coordinates. Consequently, felt accelerations that depend upon $\mathcal{V}$ will seem to the observer to be unrelated to motion; the observer will instead conclude that these terms must be forces: the fictitious and fantastical forces.

An important point is that unlike for a rigid frame, in a deforming setting the partitioning of the material acceleration $d\boldsymbol{v}/dt$ into apparent accelerations is dependent upon whether an observer chooses to record velocity and acceleration through through contravariant or covariant components. This distinction, which appears only in the fantastical force, stems from the intrinsic material derivative—compare (82) and (83). This derivative was constructed precisely to preserve the dynamics of a vector in either representation, so it usually suffices to consider only one.

b. Momentum equation in a rigidly rotating and translating frame

To motivate the development in this section we recall the standard presentation of the momentum equation in a rotating frame, as found in many textbooks, e.g. Vallis (2006, §2.1), Gill (1982, §4.5), Pedlosky (1987, §1.5–1.6), and Batchelor (2000, §3.2). See Morin (2008, §10) for a particularly thorough treatment. Although not normally done in GFD, for generality we allow the angular velocity of the moving setting to vary in time, and we also allow the moving setting to translate with a variable velocity. With these extensions, the moving setting exhausts the possibilities of rigid motion.

As before, let the y^a coordinates be embedded in an inertial frame, denoted I, that we take to be stationary. Now let the x^i coordinate system be temporarily limited to rigid motion so that it may be considered to be embedded in a rigidly rotating and translating frame of reference, denoted R. The absolute velocity $\boldsymbol{v}$ and velocity relative to the rotating frame $\boldsymbol{u}$ are then given by

$$\boldsymbol{v} = \frac{d\boldsymbol{s}}{dt}, \qquad \boldsymbol{u} = \left(\frac{d\boldsymbol{r}}{dt}\right)_R \tag{93}$$

where the meaning of the notation "$(\cdot)_R$" is that the basis vectors of the rotating frame R are to be held fixed while the derivative is carried out. Note that the material derivative with the basis vectors held fixed is equivalent to the definition $\boldsymbol{u} \equiv dx^i/dt\big|_\xi\,\boldsymbol{e}_i$ employed previously in (48).

Conservation of momentum, expressed in terms of force per unit mass, appears in the inertial frame as

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\,\boldsymbol{v} = \boldsymbol{F} \tag{94}$$

where $\boldsymbol{F}$ is a generic forcing term, the details of which are not important here. In the rotating frame this becomes

$$\left(\frac{\partial \boldsymbol{u}}{\partial t}\right)_R + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u} = \overbrace{-2\boldsymbol{\Omega}\times\boldsymbol{u}}^{\text{Coriolis}} \overbrace{-\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})}^{\text{centrifugal}} \overbrace{-\frac{d\boldsymbol{\Omega}}{dt}\times\boldsymbol{r}}^{\text{Euler}} \overbrace{-\frac{d^2\boldsymbol{S}}{dt^2}}^{\text{translational}} + \boldsymbol{F}\,, \quad (95)$$

where the first four terms on the right-hand side are *apparent* or *fictitious forces* inferred within the rotating frame.

The first two apparent forces are well known in geophysical fluid dynamics. The Coriolis force appears to accelerate the fluid perpendicular to the direction of its apparent velocity $\boldsymbol{u}$ in the rotating frame, while the centrifugal force, which appears to push fluid parcels radially outwards, is generally subsumed into the gravitational portion of $\boldsymbol{F}$. The third and fourth apparent forces are typically neglected in GFD, but we include them for completeness. The third apparent force, known as the Euler force (Marsden and Ratiu 1999, §8.6), the azimuthal force (Morin 2008, §10.2.4), or by no name (Batchelor 2000, §3.2), arises due to variability in the rotation rate Ω. We experience this in rotating amusement park rides, which apparently push us backwards as a carousel begins. It is important in planetary science problems such as tidally-locked moons. The final apparent force, the translational force, arises due to variability in the translational velocity $d\boldsymbol{S}/dt$ of the moving setting, and is the familiar "g-force" one feels inside an accelerating vehicle or elevator.

To transform the momentum equation from the inertial frame to the rotating frame, we first recognize that the absolute fluid velocity is the sum of the velocity observed in the rotating frame, the velocity induced by its rotation, and the translational velocity of the origin,

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{\Omega}\times\boldsymbol{r} + \frac{d\boldsymbol{S}}{dt}, \quad (96)$$

where the second term arises from the rate of change of the basis vectors induced by $\boldsymbol{\Omega}$, see e.g. Pedlosky (1987, §1.5) or Morin (2008, §9.5, §10.1). A second differentiation shows that accelerations in the inertial frame and the rigidly translating and rotating frame are related by

$$\frac{d\boldsymbol{v}}{dt} = \left(\frac{d\boldsymbol{u}}{dt}\right)_R + \boldsymbol{\Omega}\times\boldsymbol{u} + \boldsymbol{\Omega}\times\frac{d\boldsymbol{r}}{dt} + \frac{d\boldsymbol{\Omega}}{dt}\times\boldsymbol{r} + \frac{d^2\boldsymbol{S}}{dt^2}\,, \quad (97)$$

where again the second term arises from the rotation of the basis vectors. Noting that $\boldsymbol{r} = \boldsymbol{s} - \boldsymbol{S}$ and consequently $d\boldsymbol{r}/dt = \boldsymbol{v} - d\boldsymbol{S}/dt = \boldsymbol{u} + \boldsymbol{\Omega}\times\boldsymbol{r}$ from (96), this becomes

$$\frac{d\boldsymbol{v}}{dt} = \left(\frac{d\boldsymbol{u}}{dt}\right)_R + 2\boldsymbol{\Omega}\times\boldsymbol{u} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}) + \frac{d\boldsymbol{\Omega}}{dt}\times\boldsymbol{r} + \frac{d^2\boldsymbol{S}}{dt^2}. \quad (98)$$

Finally, the time derivative appearing on the right-hand side is expanded via the chain rule as

$$\left(\frac{d\boldsymbol{u}}{dt}\right)_R = \left(\frac{\partial \boldsymbol{u}}{\partial t}\right)_R + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u}. \quad (99)$$

Inserting these last two equations into (94) and moving terms to the right-hand-side yields (95).

c. Momentum equation in a deforming setting

Having presented the momentum equation appropriate for a rigidly moving frame, which includes familiar terms arising from uniform rotation as well as some less familiar terms, we now use the intrinsic derivative to arrive at a more general form that remains appropriate when the setting deforms. Let the x coordinates again refer to a deformable, moving setting. The momentum equation, in its broadest form is

$$\frac{d\boldsymbol{v}}{dt} = \frac{1}{\rho}\boldsymbol{F} \quad (100)$$

which we represent in component form relative to a moving setting as

$$\frac{\delta v^i}{\delta t} = \frac{1}{\rho}F^i. \quad (101)$$

Expanding using $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{\mathcal{V}}$ and collecting terms in $\boldsymbol{\mathcal{V}}$ to the right-hand side, we have

$$\left.\frac{\partial u^i}{\partial t}\right|_x + u_j\nabla^j u^i = -2u_j\nabla^j\mathcal{V}^i - \mathcal{V}_j\nabla^j\mathcal{V}^i - \left.\frac{\partial \mathcal{V}^i}{\partial t}\right|_x + \frac{1}{\rho}F^i. \quad (102)$$

Lowering an index with g_{ik} and using (90) we arrive at the covariant form

$$\left.\frac{\partial u_k}{\partial t}\right|_x + u^j\nabla_j u_k = -2u^j\nabla_{[j}\mathcal{V}_{k]} - \mathcal{V}^j\nabla_{[j}\mathcal{V}_{k]} + \mathcal{V}^j\nabla_{(j}\mathcal{V}_{k)} - \left.\frac{\partial \mathcal{V}_k}{\partial t}\right|_x + \frac{1}{\rho}F_k. \quad (103)$$

Given knowledge of the forces $\boldsymbol{F}$, these forms of the momentum are numerically implementable for any moving coordinate system: material, isopycnal, pressure, or any other convenient choice. It remains only to determine the form of the metric coefficients and the components

of $\boldsymbol{\mathcal{V}}$ for each coordinate system, which will be done explicitly in a companion paper (Feske et al. 2026). Thus (102) vastly expands the scope of possibilities over those afforded by the momentum equation in a rigid frame (95) which already has more apparent forces than are included in most GFD textbooks. The companion paper presents concrete applications of (102) to various common coordinate systems employed in geophysical fluid mechanics.

d. Interpretation of terms

In equations (102) and (103) we have a general form of the momentum equations for arbitrarily moving and deforming settings. Deformation in particular leads to a class of terms that have not previously been described in full generality. In this section we interpret those terms, propose a terminology to categorize them, and relate them to terms in the rigid frame equation (95).

Any second-order tensor can be expanded into two portions: a symmetric part (which can be made *deviatoric* by further separating its isotropic part if desired), and an antisymmetric or "skew-symmetric" part. Applying this decomposition to a generic velocity gradient,

$$\nabla_j v_i = \overbrace{\frac{1}{2}\left(\nabla_j v_i + \nabla_i v_j\right)}^{\text{symmetric\sim strain,divergence}} + \overbrace{\frac{1}{2}\left(\nabla_j v_i - \nabla_i v_j\right)}^{\text{skew/spin\sim vorticity}} \tag{104}$$

The symmetric portion is associated with strain (the deviatoric part) and divergence (the isotropic part), and the antisymmetric portion is associated with spin and vorticity. See Table 3 for a comparison of symmetry operators between symbolic and index notation. The skew-symmetric portion consists of the same information as the curl. One may write $\boldsymbol{e}^i \cdot \boldsymbol{\nabla} \times \boldsymbol{w} = \varepsilon^{ijk}\nabla_{[j}w_{k]}$, expressing the curl of any vector $\boldsymbol{w}$ in terms of the antisymmetric part of its gradient tensor.[6] The skew-symmetric velocity tensor $\nabla_{[j}v_{k]}$ corresponding to the vorticity, i.e. the curl of the velocity, is known as the *spin tensor* (Itskov 2015, p 61). It is exactly half the vorticity 2-form, which will be discussed later in section 6b. The symmetric part $\nabla_{(j}v_{k)}$ is the *strain rate tensor*. We will refer to these operations on the setting velocity as the setting spin and setting strain rate.

To related the terms in (102) and (103) back to the rigid frame equation (98), we make a small modification, adding and subtracting canceling terms, and the momentum equation (102) becomes

$$\begin{aligned}\left.\frac{\partial u^i}{\partial t}\right|_x + u^j\nabla_j u^i = &- \overbrace{2u_j\nabla^{[j}\mathcal{V}^{i]}}^{\text{generalized Coriolis}} - \overbrace{2u_j\nabla^{(j}\mathcal{V}^{i)}}^{\text{augmented Coriolis}} \\ &- \overbrace{\left(\mathcal{V}_j - \frac{\delta S_j}{\delta t}\right)\nabla^{[j}\mathcal{V}^{i]}}^{\text{generalized centrifugal}} - \overbrace{\left(\mathcal{V}_j - \frac{\delta S_j}{\delta t}\right)\nabla^{(j}\mathcal{V}^{i)}}^{\text{augmented centrifugal}} \\ &- \overbrace{\left.\frac{\partial \mathcal{V}^i}{\partial t}\right|_x - \frac{\delta S_j}{\delta t}\nabla^{[j}\mathcal{V}^{i]}}^{\text{generalized Euler-translational}} - \overbrace{\frac{\delta S_j}{\delta t}\nabla^{(j}\mathcal{V}^{i)}}^{\text{augmented translational}} \\ &+ \frac{1}{\rho}F^i.\end{aligned} \tag{105}$$

Recall that $\boldsymbol{S}$ is the vector that points from the origin of the fixed frame to the origin of the moving setting. The apparent forces become generalizations of the standard fictitious forces—Coriolis, centrifugal, and Euler and translational forces—all of which will be shown shortly to reduce to their familiar forms for the case of rigid motion. While this modification from (102) is neither unique nor necessary for numerical implementation, it is useful in comparing the momentum equation in a moving setting to its rigid-frame counterpart. The *augmented* versions of these forces all vanish under the case of rigid motion because they depend on the setting strain rate, but can contribute important additional effects under a general setting velocity. As these augmented terms are unfamiliar in any case, we combine them into the *fantastical* force, as

$$\begin{aligned}\left.\frac{\partial u^i}{\partial t}\right|_x + u^j\nabla_j u^i = &- \overbrace{2u_j\nabla^{[j}\mathcal{V}^{i]}}^{\text{generalized Coriolis}} - \overbrace{\left(\mathcal{V}_j - \frac{\delta S_j}{\delta t}\right)\nabla^{[j}\mathcal{V}^{i]}}^{\text{generalized centrifugal}} \\ &- \overbrace{\left.\frac{\partial \mathcal{V}^i}{\partial t}\right|_x - \frac{\delta S_j}{\delta t}\nabla^{[j}\mathcal{V}^{i]}}^{\text{generalized Euler-translational}} - \overbrace{\left(2u_j + \mathcal{V}_j\right)\nabla^{(j}\mathcal{V}^{i)}}^{\text{fantastical}} \\ &+ \frac{1}{\rho}F^i.\end{aligned} \tag{106}$$

Furthermore, gathering the setting spin terms and strain rate terms separately is convenient because, unlike the Coriolis- and centrifugal-like terms, the fantastical force differs in form between contravariant and covariant repre-

---

[6] This is readily found using the Hodge star operator $\star(\cdot)$ of the exterior calculus:

$$\boldsymbol{\nabla}\times\boldsymbol{\mathcal{V}} = (\star\mathrm{d}\boldsymbol{\mathcal{V}}^\flat)^\sharp = \boldsymbol{e}^i\frac{1}{2}\boldsymbol{\varepsilon}_i{}^{jk}\left(\nabla_j\mathcal{V}_k - \nabla_k\mathcal{V}_j\right) = \boldsymbol{e}_i\,\varepsilon^{ijk}\nabla_{[j}\mathcal{V}_{k]}$$

where $\mathrm{d}\boldsymbol{\mathcal{V}}^\flat$ is the exterior derivative of the 1-form associated with $\boldsymbol{\mathcal{V}}$ by the metric tensor. See, e.g., §4.2 of Marsden and Ratiu (1999), section 2.9 of Carroll (2004), section 36.5 of Needham (2021), or section 3c of Lilly et al. (2024) for further details.

sentations (Table 5).

$$\left.\frac{\partial u_k}{\partial t}\right|_x + u^j\nabla_j u_k = -\overbrace{2u^j\nabla_{[j}\mathcal{V}_{k]}}^{\text{generalized Coriolis}} - \overbrace{\left(\mathcal{V}^j - \frac{\delta S^j}{\delta t}\right)\nabla_{[j}\mathcal{V}_{k]}}^{\text{generalized centrifugal}} - \overbrace{\left.\frac{\partial \mathcal{V}_k}{\partial t}\right|_x - \frac{\delta S^j}{\delta t}\nabla_{[j}\mathcal{V}_{k]}}^{\text{generalized Euler−translational}} + \overbrace{\mathcal{V}^j\nabla_{(j}\mathcal{V}_{k)}}^{\text{fantastical}} + \frac{1}{\rho}F_i. \tag{107}$$

Table 5 presents (95) together with (106) and (107). The form of the fantastical force depends on whether the momentum is represented by a contravariant velocity as in (106) or a covariant velocity as in (107), because of (82) and (83). This is a way of saying that the term "fantastical force" does not refer to a physically real force field invariant to our choice of coordinates; we should not expect it to, given that it is an apparent force. "Fantastical" is simply a description of the terms that happen to be proportional to $\nabla_{(j}\mathcal{V}_{i)}$ in a given setting.

e. Fantastical forces vanish under rigid motion

We now show that the fantastical forces vanish in the special case of rigidly rotating and translating frames; the fictitious forces are the minimal pieces of the general equation that reduce to the familiar terms for rigid transformations, and (106) and (107) recover (95) when the moving setting does not deform. In that case, the setting velocity components are, by equations (96) and (51),

$$\mathcal{V}^i \to \mathcal{K}^i \equiv \varepsilon^{ijk}\Omega_j r_k + \frac{\delta S^i}{\delta t}, \tag{108}$$

from which it follows immediately that

$$\mathcal{K}^i - \frac{\delta S^i}{\delta t} = \varepsilon^{ijk}\Omega_j r_k = \boldsymbol{e}^i\cdot(\boldsymbol{\Omega}\times\boldsymbol{r}). \tag{109}$$

We choose the label $\mathcal{K}$ here because rigid coordinate motion is generated by *Killing vector fields*, as shown in section 4h. From (106) we see that the derivatives we need to evaluate are under rigid motions just

$$\begin{aligned}
&\nabla^j\mathcal{K}^i = \nabla^{[j}\mathcal{K}^{i]} = \varepsilon^{i\ell k}\nabla^j(\Omega_\ell r_k) = \varepsilon^{i\ell k}\delta^j_k\Omega_\ell = \varepsilon^{i\ell j}\Omega_\ell,\\
&\nabla^{(j}\mathcal{K}^{i)} = 0\,,\\
&\left.\frac{\partial\mathcal{K}^i}{\partial t}\right|_x = \left.\frac{\partial^2 S^i}{\partial t^2}\right|_x + \varepsilon^{ijk}\left.\frac{\partial}{\partial t}\right|_x\left(\Omega_j s_k\right).
\end{aligned} \tag{110}$$

The spatial derivatives of S^i and Ω_j are zero. The partial time derivative of r_k when holding x fixed is zero: all of the change in $\partial\boldsymbol{r}/\partial t|_x$ is encapsulated in the change of the basis vectors. Metrinilic and other properties from section 3b and Table A1 are also used. Shifting of indices from upper to lower position can occur later as the situation demands. Importantly, the velocity gradient tensor is antisymmetric in this case, which means that the fantastical force is zero for rigid rotation and translation in (106) and (107).

We find the apparent force terms to be

$$\begin{aligned}
-2u_j\nabla^{[j}\mathcal{K}^{i]} &= -2u_j\varepsilon^{i\ell j}\Omega_\ell = \boldsymbol{e}^i\cdot(-2\boldsymbol{\Omega}\times\boldsymbol{u}),\\
-\left(\mathcal{K}_j - \frac{\delta S_j}{\delta t}\right)\nabla^{[j}\mathcal{K}^{i]} &= -\left(\varepsilon_{jk\ell}\Omega^k r^\ell\right)\varepsilon^{imj}\Omega_m\\
&= \left(r^i\Omega^k - r^k\Omega^i\right)\Omega_k\\
&= \boldsymbol{e}^i\cdot[-\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})],\\
-\left(2u_j + \mathcal{K}_j\right)\nabla^{(j}\mathcal{K}^{i)} &= 0,\\
-\left.\frac{\partial\mathcal{K}^i}{\partial t}\right|_x - \frac{\delta S_j}{\delta t}\nabla^{[j}\mathcal{K}^{i]} &= -\varepsilon^{ijk}\frac{\delta\Omega_j}{\delta t}r_k - \frac{\delta^2 S^i}{\delta t^2}\\
&= \boldsymbol{e}^i\cdot\left(-\frac{d\boldsymbol{\Omega}}{dt}\times\boldsymbol{r} - \frac{d^2\boldsymbol{S}}{dt^2}\right),
\end{aligned} \tag{111}$$

and we recover the rigid frame transformation (95). Under arbitrary setting velocities $\boldsymbol{\mathcal{V}}$, these simplifications do not occur.

The momentum equations summarized in Table 5 show that both the Coriolis and centrifugal forces are closely related to advection terms, the former arising from the spin part of the advection of the setting velocity $\boldsymbol{\mathcal{V}}$ by the relative velocity $\boldsymbol{u}$ plus the spin part of the final term in (82), and the latter resembling self-advection of the spin portion of the setting velocity gradient, $\boldsymbol{\Omega}\times\boldsymbol{r}$.

The fantastical force terms arise, by definition, only when the setting velocity gradient tensor has a non-zero symmetric part, which doesn't occur under rigid rotation and translation. The grouping of the terms in Table 5 is useful for making the analogy to rigid frames, but it is by no means the simplest.

f. Simplified expression of the deforming momentum equations

The deforming momentum equations can be written more simply by exploiting symmetries to combine terms into more familiar forms having intuitive geometric interpretations. Complications arising from deformation become implicit within a final equation that is at once general and familiar. Combining two terms from (103) into a gradient like so

TABLE 5. The momentum equation in symbolic and index forms. The first row shows the symbolic form in a rigid frame rotating with potentially variable angular velocity $\boldsymbol{\Omega}$ and translating with velocity $d\boldsymbol{S}/dt$. The second and third rows show the index form for a general moving, deforming setting; u^i is the relative velocity in that setting. Terms in the indexed equations are grouped to reduce to the symbolic terms for a rigid frame. The indexed equations are far more general because the moving coordinate system is not constrained to be rigid, but rather moves *and* deforms according to the setting velocity field $\boldsymbol{\mathcal{V}} = \mathcal{V}^i \boldsymbol{e}_i$. Apart from the body forces $\boldsymbol{F}$, the terms on the right-hand side are apparent forces that generalize those encountered in the rigid case. When we choose the rigid setting velocity $\boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{K}} = \boldsymbol{\Omega} \times \boldsymbol{r} + d\boldsymbol{S}/dt$, the lower rows reduce to the first. Additional forces $\boldsymbol{F}$ can be applied to the right-hand side.

Equation	Tendency	Advection	Coriolis	Centrifugal	Euler & Translational	Fantastical	Forces
(95)	$\left(\frac{\partial \boldsymbol{u}}{\partial t}\right)_R$	$+(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u} =$	$-2\boldsymbol{\Omega}\times\boldsymbol{u}$	$-\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})$	$-\frac{d\boldsymbol{\Omega}}{dt}\times\boldsymbol{r} - \frac{d^2\boldsymbol{S}}{dt^2}$	$+\boldsymbol{0}$	$+\frac{1}{\rho}\boldsymbol{F}$
(107)	$\left.\frac{\partial u_k}{\partial t}\right\|_x$	$+u^j\nabla_j u_k =$	$-2u^j\nabla_{[j}\mathcal{V}_{k]}$	$-\left(\mathcal{V}^j - \frac{\delta S^j}{\delta t}\right)\nabla_{[j}\mathcal{V}_{k]}$	$-\left.\frac{\partial \mathcal{V}_k}{\partial t}\right\|_x - \frac{\delta S^j}{\delta t}\nabla_{[j}\mathcal{V}_{k]}$	$+\mathcal{V}^j\nabla_{(j}\mathcal{V}_{k)}$	$+\frac{1}{\rho}F_k$
(106)	$\left.\frac{\partial u^i}{\partial t}\right\|_x$	$+u^j\nabla_j u^i =$	$-2u_j\nabla^{[j}\mathcal{V}^{i]}$	$-\left(\mathcal{V}_j - \frac{\delta S_j}{\delta t}\right)\nabla^{[j}\mathcal{V}^{i]}$	$-\left.\frac{\partial \mathcal{V}^i}{\partial t}\right\|_x - \frac{\delta S_j}{\delta t}\nabla^{[j}\mathcal{V}^{i]}$	$-\left(2u_j + \mathcal{V}_j\right)\nabla^{(j}\mathcal{V}^{i)}$	$+\frac{1}{\rho}F^i$

$$
\begin{aligned}
-\mathcal{V}^j\nabla_{[j}\mathcal{V}_{k]} + \mathcal{V}^j\nabla_{(j}\mathcal{V}_{k)} &= \mathcal{V}^j\nabla_{[k}\mathcal{V}_{j]} + \mathcal{V}^j\nabla_{(k}\mathcal{V}_{j)} \\
&= \mathcal{V}^j\nabla_k\mathcal{V}_j \\
&= \frac{1}{2}\left(\mathcal{V}^j\nabla_k\mathcal{V}_j + \mathcal{V}_j\nabla_k\mathcal{V}^j\right) \\
&= \frac{1}{2}\nabla_k\left(\mathcal{V}_j\mathcal{V}^j\right),
\end{aligned}
\tag{112}
$$

we see that the quantity $\mathcal{V}^j\left(\nabla_{(j}\mathcal{V}_{k)} - \nabla_{[j}\mathcal{V}_{k]}\right)$ in the covariant representation can always be written as minus the gradient of a generalized setting potential

$$
\Phi = \frac{-\mathcal{V}_j\mathcal{V}^j}{2}. \tag{113}
$$

In the case of rigid rotation $\boldsymbol{\mathcal{V}} = \boldsymbol{\Omega}\times\boldsymbol{r}$ about the origin, (113) reproduces the standard centrifugal potential—see for example Vallis (2017) (2.14). Equation (103) becomes

$$
\left.\frac{\partial u_k}{\partial t}\right|_x + u^j\nabla_j u_k = -2u^j\nabla_{[j}\mathcal{V}_{k]} - \nabla_k\Phi - \left.\frac{\partial \mathcal{V}_k}{\partial t}\right|_x + \frac{1}{\rho}F_k. \tag{114}
$$

Raising the index and moving apparent forces to the right-hand side gives

$$
\begin{aligned}
\left.\frac{\partial u^i}{\partial t}\right|_x + u_j\nabla^j u^i = -2u_j\nabla^{[j}\mathcal{V}^{i]} - \nabla^i\Phi - \left.\frac{\partial \mathcal{V}^i}{\partial t}\right|_x \\
- 2u_j\nabla^{(j}\mathcal{V}^{i)} + \frac{1}{\rho}F^i.
\end{aligned}
\tag{115}
$$

The non-Coriolis apparent forces in both equations sum to the tendency of $\boldsymbol{\mathcal{V}}$, that is, we have

$$
\begin{aligned}
\left.\frac{\partial \boldsymbol{\mathcal{V}}}{\partial t}\right|_x &= \boldsymbol{e}^k\left(\left.\frac{\partial \mathcal{V}_k}{\partial t}\right|_x + \nabla_k\Phi\right) \\
&= \boldsymbol{e}_i\left(\left.\frac{\partial \mathcal{V}^i}{\partial t}\right|_x + \nabla^i\Phi + 2\mathcal{V}_j\nabla^{(j}\mathcal{V}^{i)}\right)
\end{aligned}
\tag{116}
$$

A natural grouping of terms appears in Table 6, where the entries in each column are related by index raising and lowering. This form shows most clearly the the difference in fantastical force terms between the two representations of the momentum equation comes from raising and lowering the indices on $\boldsymbol{u}$ and $\boldsymbol{\mathcal{V}}$ through the partial time derivative, which is not metric compatible and introduces fantastical force terms via equation (90) for non-rigid settings.

Equation (114) is the simplest form of the generalized momentum equation. Term-by-term it looks similar to the form we are used to from the rotating frame, but it has been carefully constructed to include the effects of arbitrary setting deformation. Note that the covariant expression of the momentum equation is simplest; a full explanation of this is out of scope, but the geometric mechanics literature provides insight (Holm et al. 1998). This asymmetry between the covariant and contravariant representations appears in earlier work from the continuum mechanics literature following the work of Oldroyd (1950), whose lower-convected and upper-convected derivatives characterize Oldroyd A fluids and Oldroyd B fluids, discussed in Hinch and Harlen (2021).

g. Contravariant versus covariant representations

Our discussion of the fantastical forces has repeatedly mentioned that the momentum equation in a moving setting—but not the resulting dynamics—depends upon

TABLE 6. The covariant and contravariant versions of the momentum equation (114) and (115), organized so that the terms in each column are related by index raising or lowering. This shows that the differences between the covariant and contravariant representations of the momentum equation result when we pass the metric components through the partial time derivative to raise the index.

Equation	Tendency/Fantastical	Advection		Coriolis	Setting Potential	Setting Tendency/Fantastical	Forces
(114)	$\left.\frac{\partial u_k}{\partial t}\right\|_x$	$+u^j\nabla_j u_k$	$=$	$-2u^j\nabla_{[j}\mathcal{V}_{k]}$	$-\nabla_k\Phi$	$-\left.\frac{\partial \mathcal{V}_k}{\partial t}\right\|_x$	$+\frac{1}{\rho}F_k$
(115)	$\left.\frac{\partial u^i}{\partial t}\right\|_x + 2u_j\nabla^{(j}\mathcal{V}^{i)}$	$+u^j\nabla_j u^i$	$=$	$-2u_j\nabla^{[j}\mathcal{V}^{i]}$	$-\nabla^i\Phi$	$-\left.\frac{\partial \mathcal{V}^i}{\partial t}\right\|_x - 2\mathcal{V}_j\nabla^{(j}\mathcal{V}^{i)}$	$+\frac{1}{\rho}F^i$

the choice of a contravariant or covariant representation of momentum. To see this another way, we borrow a concept from continuum mechanics: the *upper* and *lower convected time derivatives*, written respectively as

$$\overset{\triangledown}{\boldsymbol{w}} = \frac{d\boldsymbol{w}}{dt} - \boldsymbol{w}\left(\boldsymbol{\nabla}\otimes\boldsymbol{v}\right) \tag{117}$$

$$\overset{\triangle}{\boldsymbol{w}} = \frac{d\boldsymbol{w}}{dt} + \boldsymbol{w}\left(\boldsymbol{\nabla}\otimes\boldsymbol{v}\right)^T \tag{118}$$

These derive ultimately from Oldroyd (1950), and give the apparent rates of change of the vector $\boldsymbol{w}$ observed when moving at the fluid velocity $\boldsymbol{v}$. The triangles "$\triangledown$" and "$\triangle$" indicate the valence of the vector components to which the partial time derivative is applied, the former applies when measuring w^i or w^a, and the latter when measuring w_i or w_a. This notation is strange because, though it presents as an operation on a tensor $\boldsymbol{w}$, its form depends on the component representation of that tensor. Nonetheless, it is common in continuum mechanics (e.g., Dimitrienko 2011; Hinch and Harlen 2021). See Appendix B for an explicit, *component*-wise definition of these derivatives applicable to arbitrary tensor fields. Rearranging, we can write

$$\frac{d\boldsymbol{w}}{dt} = \overset{\triangledown}{\boldsymbol{w}} + \boldsymbol{w}\left(\boldsymbol{\nabla}\otimes\boldsymbol{v}\right) \tag{119}$$

$$\frac{d\boldsymbol{w}}{dt} = \overset{\triangle}{\boldsymbol{w}} - \boldsymbol{w}\left(\boldsymbol{\nabla}\otimes\boldsymbol{v}\right)^T \tag{120}$$

Because these derivatives deal particularly with convected coordinates (which we would call *advected* or Lagrangian coordinates), we must make the particular choice $\boldsymbol{\mathcal{V}} = \boldsymbol{v}$.

When these operators are applied to the local velocity $\boldsymbol{u}$, this presentation shows clearly that, depending on which set of components we measure, there are two possible partitionings of the material acceleration $d\boldsymbol{u}/dt$ into an apparent local rate of change—the first terms on the right hand side—and apparent forces, the negative of the second term. This choice is arbitrary, and does not affect the actual dynamics. The apparent forces are the same if and only if the transformation is rigid, that is, the gradient tensor of the setting velocity $\boldsymbol{\nabla}\otimes\boldsymbol{v}$ is everywhere antisymmetric. This is why we distinguish the fictitious forces (which are constructed from only the setting spin part) from the fantastical forces (which are constructed from the setting strain rate and divergence) as shown in Table 5.

The apparent forces arising in rigidly moving frames, being independent of the valence of the tensor's representation, are of a different nature than those which arise more generally in the presence of deformation. To understand this, consider that a moving observer will feel their total acceleration $d\boldsymbol{v}/dt$, and will not be able to viscerally distinguish between its various components. An observer standing in a rotating room, however, feels an outward acceleration, takes note of the fact that they are not in motion relative to the room, and infers the presence of a force acting outwards. In this way apparent forces are inferred from felt accelerations and measured motion together. The above result implies that when the setting is not rigid, the inference of apparent forces from felt accelerations no longer has a unique result. See Appendix C for a worked example showing how the apparent forces appear in an expanding coordinate system. Their character is indeed quite different between the covariant and contravariant representations.

6. Applications of the intrinsic derivative to the vorticity equation

a. The Cauchy vorticity formula

The intrinsic derivative provides a straightforward, concise derivation of Cauchy's frozen-in vorticity result (Cauchy 1816) from the evolution of the vorticity per unit density as it appears in Aris (1962) equation (4.5.5):

$$\frac{d}{dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \left(\frac{\boldsymbol{\omega}}{\rho}\cdot\boldsymbol{\nabla}\right)\boldsymbol{v} + \frac{1}{\rho}\boldsymbol{\nabla}\times\boldsymbol{a} \tag{121}$$

in a fixed frame where $\boldsymbol{a}$ is the acceleration (that is, force per unit mass). If we take the barotropic, conservative case where $\boldsymbol{\nabla}\times\boldsymbol{a} = \boldsymbol{\nabla}\times\boldsymbol{F}/\rho = 0$ and convert to index notation this becomes

$$\frac{\delta}{\delta t}\left(\frac{\omega^a}{\rho}\right) = \frac{\omega^b}{\rho}\nabla_b v^a \tag{122}$$

In any moving setting, the equation is just the same—this is precisely the point of the intrinsic derivative:

$$\frac{\delta}{\delta t}\left(\frac{\omega^i}{\rho}\right) = \frac{\omega^j}{\rho}\nabla_j v^i \tag{123}$$

In the Lagrangian setting we have $\boldsymbol{\mathcal{V}} = \boldsymbol{v}$ and $\boldsymbol{u} = \mathbf{0}$. Along with equation (82) this gives

$$\left.\frac{\partial}{\partial t}\right|_x \left(\frac{\omega^i}{\rho}\right) = 0 \tag{124}$$

That is, the Lagrangian components of vorticity per unit density do not change for a given fluid parcel. This result is usually expressed in terms of the Jacobian transformation between the Eulerian and Lagrangian settings, for example in Frisch and Villone (2014) equations (9) and (10). Here, by contrast, we avoid explicit manipulation of Jacobians because the kinematic relationship of the fixed and Lagrangian settings is built into the definition of the intrinsic derivative.

b. Lamb form and the vorticity equation

We now derive the vorticity equation directly, using the Lamb form of the momentum equation as a stepping stone. Applying the same property from (112) again to the apparent velocity gradient, we can rewrite $u^j\nabla_j u_k$ as

$$u^j\nabla_j u_k = \frac{1}{2}\nabla_k(u_j u^j) + 2u^j\nabla_{[j}u_{k]} \tag{125}$$

and (114) becomes

$$\left.\frac{\partial v_k}{\partial t}\right|_x + 2u^j\nabla_{[j}v_{k]} = -\frac{1}{2}\nabla_k(u_j u^j - \mathcal{V}_j\mathcal{V}^j) + \frac{1}{\rho}F_k. \tag{126}$$

which is the Lamb form of the momentum equation, so called because the second term on the right-hand side of (125) are minus the covariant components of the Lamb vector $\boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{u})$ (Lamb 1895). $2\nabla_{[j}v_{k]}$ are the components of the the vorticity 2-form ω

$$\omega_{jk} \equiv 2\nabla_{[j}v_{k]} \tag{127}$$

This is twice the spin tensor of Itskov (2015). Note that the form (126) of the momentum equation is expressed entirely in terms of differential forms[7]. Readers may be more familiar with the vorticity *vector* $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{v} = \omega^i\boldsymbol{e}_i$; these two objects can be placed into one-to-one correspondence by applying the Hodge star operator and then raising the index[8]:

$$\frac{1}{2}\varepsilon^{ijk}\omega_{jk} = \omega^i \quad\longleftrightarrow\quad \omega_{jk} = \varepsilon_{jki}\omega^i, \tag{128}$$

The decomposition $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{\mathcal{V}}$ lets us break the total vorticity into the setting and relative vorticities

$$\zeta_{jk} \equiv 2\nabla_{[j}u_{k]} \quad\text{and}\quad f_{jk} \equiv 2\nabla_{[j}\mathcal{V}_{k]}, \tag{129}$$

and a similar decomposition exists for the vorticity vector. Given rigid planetary rotation $\boldsymbol{\mathcal{V}} = \boldsymbol{\Omega}\times\boldsymbol{r}$ about the origin, the setting vorticity is the planetary vorticity.

Having packed as much as we can inside the gradient on the right-hand side of (126), we can derive the vorticity equation in the usual way by taking the curl of both sides. It is helpful to note that, due to the antisymmetry of the Levi-Civita tensor and the symmetry of the Christoffel symbol, we may render the curl as either $\varepsilon^{i\ell k}\nabla_\ell$ or $\varepsilon^{i\ell k}\partial_\ell$. With the help of (128) and Table A2 we arrive at

$$\left(\left.\frac{\partial}{\partial t}\right|_x + \mathcal{L}_{\boldsymbol{u}}\right)\omega^i = -\omega^i\nabla_k v^k + \frac{1}{\rho}(\boldsymbol{\nabla}\times\boldsymbol{F})^i \tag{130}$$

and using (82) this is equivalent to

$$\frac{\delta\omega^i}{\delta t} = \omega^k\nabla_k v^i - \omega^i\nabla_k v^k + \frac{1}{\rho}(\boldsymbol{\nabla}\times\boldsymbol{F})^i, \tag{131}$$

which is the standard form of the vorticity equation for a compressible fluid, for example in Kundu et al. (2016) (5.13) or Thorne and Blandford (2017) (14.5).

c. The Cauchy vorticity formula, again

We now revisit Cauchy's frozen-in vorticity result, this time deriving it directly from (130). For an inviscid, barotropic fluid we have $\boldsymbol{\nabla}\times\boldsymbol{F} = \mathbf{0}$. In the Lagrangian setting $\boldsymbol{\mathcal{V}} = \boldsymbol{v}$ and $\boldsymbol{u} = \mathbf{0}$, so the vorticity equation becomes

$$\left.\frac{\partial\omega^i}{\partial t}\right|_x = -\omega^i\nabla_k v^k, \tag{132}$$

the continuity equation is

[7] A differential form is simply an entirely antisymmetric, covariant tensor. It is common to denote these with lower-case Greek letters as the base symbol. Because the antisymmetry is understood, the brackets are omitted. Forms are a proper subset of general tensors that allow pleasantly geometric expressions of physical equations, and they have their own self-contained calculus—the exterior calculus of differential forms. See Needham (2021) Part 5 for a particularly good exposition of this topic and its great utility in physics.

[8] Readers familiar with the calculus of forms might prefer the index-free version of equation (128): $(\star\omega)^\sharp = \boldsymbol{\omega} \quad\longleftrightarrow\quad \omega = \star(\boldsymbol{\omega}^\flat)$

$$\left.\frac{\partial \rho}{\partial t}\right|_x = -\rho \nabla_k v^k, \tag{133}$$

and therefore we have

$$\begin{aligned}\left.\frac{\partial}{\partial t}\right|_x \left(\frac{\omega^i}{\rho}\right) &= \frac{1}{\rho^2}\left(\rho \left.\frac{\partial \omega^i}{\partial t}\right|_x - \omega^i \left.\frac{\partial \rho}{\partial t}\right|_x\right) \\ &= \frac{\rho\omega^i - \rho\omega^i}{\rho^2}\nabla_k v^k \\ &= 0\end{aligned} \tag{134}$$

Once again we have recovered this result in only a few lines of algebra.

7. Discussion and Conclusions

In this paper, we have developed a tensor-based framework for describing geophysical fluid dynamics in general, deforming coordinate systems. Beginning from first principles in tensor analysis, we introduced a generalized intrinsic material derivative that captures the evolution of tensor components in curvilinear bases that vary arbitrarily, but continuously, in space and time. This derivative ensures consistency across covariant and contravariant representations and across inertial and non-inertial frames, providing a unified treatment of the fundamental building blocks of GFD.

The generalized intrinsic time derivative $\delta/\delta t|^{\chi}$, taking its most general form in equation (84), allows us to calculate the total derivative of arbitrary, space-time dependent tensor fields along any parameterized path $\chi(t)$—an integral path through an arbitrary vector field $\boldsymbol{q}$—in coordinates being deformed by an arbitrary vector field $\boldsymbol{\mathcal{V}}$. If we set $\boldsymbol{\mathcal{V}} = 0$, the generalized derivative reduces to the intrinsic time derivative discussed by Grinfeld (2013) and Aris (1962). The only constraint on $\boldsymbol{\mathcal{V}}$ is that it generate an invertible, continuously differentiable transformation of the moving coordinates x^i—that is, that it generate a *diffeomorphism* on Euclidean space. In the particular case where $\boldsymbol{q} = \boldsymbol{v}$, the fluid velocity, the intrinsic derivative reduces to the intrinsic material derivative $\delta/\delta t$.

All coordinate systems—all settings—typically used in GFD are a diffeomorphism away from an inertial, Cartesian frame. These include rotating, accelerating, neutral buoyancy, terrain-following, pressure, tracer, Lagrangian, and generalized vertical coordinates. Taking into account the implications of such a coordinate transformation, we have found the general form for the apparent forces that correct for the non-inertial aspects of these coordinate systems without approximation.

We applied this generalized intrinsic derivative and derived explicit expressions for all apparent accelerations arising in GFD settings. These include the familiar fictitious forces—Coriolis, centrifugal, translational, and Euler forces—as well as the additional family of terms that arise when the coordinate system is deforming, which we have termed the *fantastical* forces. Unlike fictitious forces, the fantastical forces depend on the symmetric part of the setting-velocity gradient and therefore vanish only when the setting velocity is a Killing vector field, that is, for diffeomorphisms connecting rigid frames. Their dependence on tensor valence highlights that deformation introduces qualitatively new geometric structure into the apparent dynamics.

We further developed these general equations to pack the fantastical contributions into geometrically legible quantities: the generalized Coriolis force and the tendency of the setting velocity $\boldsymbol{\mathcal{V}}$. A part of the latter is always conservative, and we can calculate it as the negative gradient of the setting potential Φ. Finally we derive the vorticity equation, and show that its representation in terms of the intrinsic derivative recovers Cauchy's vorticity formula in just a few lines.

Our results clarify the relationship between classical rotating-frame dynamics and their fully general counterparts, showing that the standard GFD equations emerge as a special case of a broader tensorial formulation. By expressing all results using index notation, we provide equations that can be directly translated into numerical model implementations, particularly those with vertical remapping, adaptive, or arbitrarily Lagrangian-Eulerian (ALE) coordinate systems.

The following statements are all logically equivalent:

1. The moving setting x is a rigid frame,
2. $\boldsymbol{\nabla} \otimes \boldsymbol{\mathcal{V}}$ is antisymmetric,
3. $\mathcal{L}_{\boldsymbol{\mathcal{V}}} g_{ij} = 0$
4. $\left.\frac{\partial g_{ij}}{\partial t}\right|_x = 0$,
5. $\boldsymbol{\mathcal{V}}$ is a Killing vector field,
6. Oldroyd's upper and lower convected derivatives are equivalent,
7. The fantastical forces are zero.

In many GFD settings, these statements do not apply, and thus nonzero fantastical forces will arise. Although they may be small, they preserve exact symmetries and conservation laws under all coordinate changes.

The framework developed here lays the mathematical foundation for companion papers of this series, which will: extend these ideas to the full oceanic primitive equations including coordinate-agnostic specifications of common parameterizations and approximations; specify metric coefficients of common coordinate systems for GFD;

determine coordinate-independent averaging and coarse-graining operators; provide an asymptotic scaling for the fantastical forces; and derive budgets for common higher-order invariants (potential vorticity, energy). Together, these works aim to establish a unified, tensorial approach for next-generation ocean models, enabling more accurate representations of the physics of rotating, stratified flows in complex, time-varying geometries.

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Data availability statement. No data were used in this paper.

APPENDIX A

The metrinilic property for Jacobians

All the calculations here take place at fixed time, so for this purpose y^a and x^i simply represent two curvilinear coordinate systems; rules proven here hold for objects indexed in either $\{a,b,c\}$ or $\{i,j,k\}$. Tensor derivatives—that is, derivatives that act on tensor components to produce tensor components—should null out the Jacobian coefficients. Any such derivative D by definition has the property that $J^a_i D(w^i) = D(w^a)$. This is the transformation law for tensor components, and is equivalent to the statement $D(J^a_i) = 0$:

$$D(w^a) = D(J^a_i w^i) = J^a_i D(w^i). \tag{A1}$$

In this section we show explicitly that both the covariant and Lie derivatives have this property. First we note that the basis vectors $\boldsymbol{e}_i$ are in fact covariant components of a tensor as the notation suggests: by the chain rule they transform according to the tensor transformation law $\boldsymbol{e}_a = J^i_a \boldsymbol{e}_i$, see equation (27). More generally, a second order tensor $\boldsymbol{A}$ has components A^{ij} so that

$$\boldsymbol{A} = A^{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j = \boldsymbol{e}_i \otimes A^{ij}\boldsymbol{e}_j \equiv \boldsymbol{e}_i \otimes \boldsymbol{A}^i \tag{A2}$$

defining the covariant vector components $\boldsymbol{A}^i$ of $\boldsymbol{A}$, which by similar reasoning has covariant vector components $\boldsymbol{A}_i$.[9]

[9] We can see that these obey the tensor transformation law:

$$\boldsymbol{A}^a \equiv A^{ab}\boldsymbol{e}_b = J^a_i A^{ib} J^j_b \boldsymbol{e}_j = J^a_i A^{ij}\boldsymbol{e}_j \equiv J^a_i \boldsymbol{A}^i \tag{A3}$$

which makes use of the chain rule relation (27).

By the same reasoning we can now see that the basis vectors $\boldsymbol{e}^i$ and $\boldsymbol{e}_i$ are the contravariant and covariant components of the identity $\boldsymbol{I}$ because $\boldsymbol{I} = \delta^i_j \boldsymbol{e}_i \otimes \boldsymbol{e}^j$, analogous to the final equality of (A2).

$$\boldsymbol{I} = \delta^i_j \boldsymbol{e}_i \otimes \boldsymbol{e}^j = \boldsymbol{e}_i \otimes \boldsymbol{e}^i \tag{A4}$$

Now we can be confident that applying index-based tensor derivatives to the basis vectors is well-defined

a. The covariant derivative

We find that

$$\nabla_j \boldsymbol{e}_i = \partial_i \boldsymbol{e}_j - \Gamma^k{}_{ij}\boldsymbol{e}_k = \Gamma^k{}_{ij}\boldsymbol{e}_k - \Gamma^k{}_{ij}\boldsymbol{e}_k = 0. \tag{A5}$$

where we have used equation (4) in passing from the second expression to the third. This not coincidence—the covariant derivative and Christoffel symbols are defined precisely to have this property. This means that we can freely write $\nabla_i \boldsymbol{a} \equiv \partial_i \boldsymbol{a}$, and we can now show that the covariant derivative satisfies the product rule across dot products of vectors:

$$\begin{aligned}
\nabla_i(\boldsymbol{a}\cdot\boldsymbol{b}) &= \nabla_i\left(a_j b^j\right) \\
&= a_j \nabla_i b^j + b^j \nabla_i a_j \\
&= \boldsymbol{a}\cdot\boldsymbol{e}_j \nabla_i b^j + \boldsymbol{b}\cdot\boldsymbol{e}^j \nabla_i a_j \\
&= \boldsymbol{a}\cdot\nabla_i\boldsymbol{b} + \nabla_i\boldsymbol{a}\cdot\boldsymbol{b}.
\end{aligned} \tag{A6}$$

We also have

$$\nabla_j \boldsymbol{e}_a = J^b_j \nabla_b \boldsymbol{e}_a = 0 \tag{A7}$$

and therefore

$$\nabla_j J^i_a = \nabla_j(\boldsymbol{e}^i\cdot\boldsymbol{e}_a) = \nabla_j\boldsymbol{e}^i\cdot\boldsymbol{e}_a + \boldsymbol{e}^i\cdot\nabla_j\boldsymbol{e}_a = 0 \tag{A8}$$

as expected, and analogous reasoning holds for $\nabla_j J^a_i$.

b. The Lie derivative

Recalling that we can define the Lie derivative in terms of the covariant derivative, and that the Lie derivative on invariant fields is simply the directional derivative $\mathcal{L}_{\mathcal{V}} f = \mathcal{V}^i \nabla_i f$, we find that that it also satisfies the product rule over the dot product:

$$\begin{aligned}
\mathcal{L}_{\mathcal{V}}(\boldsymbol{a}\cdot\boldsymbol{b}) &= \mathcal{L}_{\mathcal{V}}\left(a_j b^j\right) \\
&= a_j \mathcal{L}_{\mathcal{V}} b^j + b^j \mathcal{L}_{\mathcal{V}} a_j \\
&= a_j\left(\mathcal{V}^k\nabla_k b^j - \cancel{b^k \nabla_k \mathcal{V}^j}\right) \\
&\qquad + b^j\left(\mathcal{V}^k\nabla_k a_j + \cancel{a_k \nabla_j \mathcal{V}^k}\right) \\
&= \boldsymbol{a}\cdot\boldsymbol{e}_j\mathcal{V}^k\nabla_k b^j + \boldsymbol{b}\cdot\boldsymbol{e}^j\mathcal{V}^k\nabla_k a_j \\
&= \boldsymbol{a}\cdot\mathcal{V}^k\nabla_k\boldsymbol{b} + \boldsymbol{b}\cdot\mathcal{V}^k\nabla_k\boldsymbol{a} \\
&= \boldsymbol{a}\cdot\mathcal{L}_{\mathcal{V}}\boldsymbol{b} + \mathcal{L}_{\mathcal{V}}\boldsymbol{a}\cdot\boldsymbol{b}
\end{aligned} \tag{A9}$$

Further, we can calculate

$$\mathcal{L}_{\mathcal{V}} \boldsymbol{e}^i = \mathcal{V}^j \nabla_j \boldsymbol{e}^i - \boldsymbol{e}^j \nabla_j \mathcal{V}^i = -\boldsymbol{e}^j \nabla_j \mathcal{V}^i \tag{A10}$$

$$\mathcal{L}_{\mathcal{V}} \boldsymbol{e}_a = \mathcal{V}^b \nabla_b \boldsymbol{e}_a + \boldsymbol{e}_b \nabla_a \mathcal{V}^b = \boldsymbol{e}_b \nabla_a \mathcal{V}^b \tag{A11}$$

so therefore

$$\begin{aligned} \mathcal{L}_{\mathcal{V}} J^i_a &= \mathcal{L}_{\mathcal{V}} (\boldsymbol{e}^i \cdot \boldsymbol{e}_a) \\ &= \mathcal{L}_{\mathcal{V}} \boldsymbol{e}^i \cdot \boldsymbol{e}_a + \boldsymbol{e}^i \cdot \mathcal{L}_{\mathcal{V}} \boldsymbol{e}_a \\ &= -\boldsymbol{e}_a \cdot \boldsymbol{e}^j \nabla_j \mathcal{V}^i + \boldsymbol{e}^i \cdot \boldsymbol{e}_b \nabla_a \mathcal{V}^b \\ &= -J^j_a \nabla_j \mathcal{V}^i + J^i_b \nabla_a \mathcal{V}^b \\ &= -\nabla_a \mathcal{V}^i + \nabla_a \mathcal{V}^i \\ &= 0 \end{aligned} \tag{A12}$$

as expected, and once again analogous reasoning holds for $\mathcal{L}_{\mathcal{V}} J^a_i$. Derivatives of all useful metric-related objects can be calculated using this strategy of decomposing components of identity into dot products of basis vectors and applying the product rule. The results of all such calculations are compiled in Table A1. The same derivatives of the components of the Levi-Civita tensor appear in Table A2. Space-time derivative commutators as applied to vector components appear in Table A3.

APPENDIX B

Oldroyd and the Intrinsic Time Derivative

The generalized intrinsic derivative lets us easily derive the expression for the convected derivative of Oldroyd (1950). Oldroyd is concerned with materially advected coordinates, and so in the following we choose the specific $\mathcal{V} = \boldsymbol{v}$. In the spirit of the rest of this paper, we prefer his original, indicial definition. This definition is preferable also because it makes clear that the upper and lower convected derivatives are not, in fact, two different derivatives of tensors, but the same derivative applied respectively to contravariant and covariant tensor *components*, in precisely the same sense as the Lie derivative or the covariant derivative. For brevity's sake, we show the derivation for the derivative as applied to vector components, however it generalizes immediately to tensors of any order. As described in the paragraph preceding Oldroyd's equation (23), the convected derivative of a vector $\boldsymbol{w} = w^i \boldsymbol{e}_i$ is nothing but the vector with components $\partial w^i / \partial t|_x$ in the moving setting. The question is: what are the inertial frame components of this vector? We treat these as tensor components by assumption, and so they transform with the Jacobian:

$$\begin{aligned} \frac{\mathfrak{d} w^a}{\mathfrak{d} t} &\equiv J^a_i \left.\frac{\partial w^i}{\partial t}\right|_x \\ &= J^a_i \left(\frac{\delta w^i}{\delta t} - v^k \nabla_k w^i + \mathcal{L}_{\boldsymbol{v}} w^i \right) \\ &= \frac{\delta w^a}{\delta t} - v^c \nabla_c w^a + \mathcal{L}_{\boldsymbol{v}} w^a \\ &= \left.\frac{\partial w^a}{\partial t}\right|_y + \mathcal{L}_{\boldsymbol{v}} w^a. \end{aligned} \tag{B1}$$

And there we have it—we have recovered Oldroyd's equation (23)[10]. In operator form, the convected derivative is therefore

$$\frac{\mathfrak{d}}{\mathfrak{d} t} = \left(\frac{\delta}{\delta t} - \nabla_{\boldsymbol{v}} + \mathcal{L}_{\boldsymbol{v}} \right) \tag{B2}$$

and this can be applied either to inertial frame components:

$$\frac{\mathfrak{d} w^a}{\mathfrak{d} t} = \left(\frac{\delta}{\delta t} - v^c \nabla_c + \mathcal{L}_{\boldsymbol{v}} \right) w^a = \left(\left.\frac{\partial}{\partial t}\right|_y + \mathcal{L}_{\boldsymbol{v}} \right) w^a \tag{B3}$$

or to moving setting components:

$$\frac{\mathfrak{d} w^i}{\mathfrak{d} t} = \left(\frac{\delta}{\delta t} - v^k \nabla_k + \mathcal{L}_{\boldsymbol{v}} \right) w^i = \left.\frac{\partial w^i}{\partial t}\right|_x \tag{B4}$$

Later, coordinate-free definitions of the convected derivative, for example in Hinch and Harlen (2021), are odd because, though superficially coordinate-free, they nevertheless depend on the valence of the underlying tensor components. The "upper convected" and "lower convected" derivatives can respectively be written in terms of Oldroyd's component-wise derivative $\mathfrak{d}/\mathfrak{d}t$ like so:

$$\begin{aligned} \overset{\triangledown}{\boldsymbol{w}} &= \boldsymbol{e}_a \frac{\mathfrak{d} w^a}{\mathfrak{d} t} = \boldsymbol{e}_i \frac{\mathfrak{d} w^i}{\mathfrak{d} t} \\ \overset{\triangle}{\boldsymbol{w}} &= \boldsymbol{e}^a \frac{\mathfrak{d} w_a}{\mathfrak{d} t} = \boldsymbol{e}^i \frac{\mathfrak{d} w_i}{\mathfrak{d} t} \end{aligned} \tag{B5}$$

Notice the similarity in form here to equations (62) and (77). These two quantities are not equal because the Lie derivative, and therefore also Oldroyd's, acts on the metric coefficients to produce a non-zero tensor:

$$\frac{\mathfrak{d} g_{ab}}{\mathfrak{d} t} = \mathcal{L}_{\boldsymbol{v}} g_{ab} = 2 \nabla_{(a} v_{b)} \tag{B6}$$

and therefore does not commute with index juggling. We find that

$$\overset{\triangle}{\boldsymbol{w}} - \overset{\triangledown}{\boldsymbol{w}} = 2 \boldsymbol{e}^a w^b \nabla_{(a} v_{b)} = \boldsymbol{e}^a w^b \mathcal{L}_{\boldsymbol{v}} g_{ab}, \tag{B7}$$

[10] His formulation includes extra terms to handle the differentiation of tensor densities, all of which are zero for tensors.

TABLE A1. A table of derivatives of the various components of the identity $\boldsymbol{I}$. Each can be derived by representing components of $\boldsymbol{I}$ as dot products of basis vectors, themselves vector-valued tensor components, and applying the product rule.

Derivative	δ^i_j	δ^a_b	g_{ij}	g_{ab}	$\boldsymbol{e}_i$	$\boldsymbol{e}^i$	$\boldsymbol{e}_a$	$\boldsymbol{e}^a$	J^i_a	J^a_i
∂_k	0	0	$2\Gamma_{(i,j)k}$	$2\Gamma_{(a,b)c}J^c_k$	$\boldsymbol{e}_j\Gamma^j{}_{ik}$	$-\boldsymbol{e}^j\Gamma^i{}_{jk}$	$\boldsymbol{e}_b\Gamma^b{}_{ac}J^c_k$	$-\boldsymbol{e}^b\Gamma^a{}_{bc}J^c_k$	$J^i_{ac}J^c_k$	J^a_{ik}
∂_c	0	0	$2\Gamma_{(i,j)k}J^k_c$	$2\Gamma_{(a,b)c}$	$\boldsymbol{e}_j\Gamma^j{}_{ik}J^k_c$	$-\boldsymbol{e}^j\Gamma^i{}_{jk}J^k_c$	$\boldsymbol{e}_b\Gamma^b{}_{ac}$	$-\boldsymbol{e}^b\Gamma^a{}_{bc}$	J^i_{ac}	$J^a_{ik}J^k_c$
$\frac{\partial}{\partial t}\big\vert_x$	0	0	$2\nabla_{(i}\mathcal{V}_{j)}$	$\mathcal{V}^c\partial_c g_{ab}$	$\boldsymbol{e}_j\nabla_i\mathcal{V}^j$	$-\boldsymbol{e}^j\nabla_j\mathcal{V}^i$	$\boldsymbol{e}_c\Gamma^c{}_{ab}\mathcal{V}^b$	$-\boldsymbol{e}^b\Gamma^a{}_{bc}\mathcal{V}^c$	$-J^i_b\partial_a\mathcal{V}^b$	$J^b_i\partial_b\mathcal{V}^a$
$\frac{\partial}{\partial t}\big\vert_y$	0	0	$2g_{k(i}\partial_{j)}\mathcal{V}^k$	0	$\boldsymbol{e}_j\partial_i\mathcal{V}^j$	$-\boldsymbol{e}^j\partial_j\mathcal{V}^i$	0	0	$-J^j_a\partial_j\mathcal{V}^i$	$J^a_j\partial_i\mathcal{V}^j$
∇_k	0	0	0	0	0	0	0	0	0	0
∇_c	0	0	0	0	0	0	0	0	0	0
$\mathcal{L}_{\mathcal{V}}$	0	0	$2\nabla_{(i}\mathcal{V}_{j)}$	$2\nabla_{(a}\mathcal{V}_{b)}$	$\boldsymbol{e}_j\nabla_i\mathcal{V}^j$	$-\boldsymbol{e}^j\nabla_j\mathcal{V}^i$	$\boldsymbol{e}_b\nabla_a\mathcal{V}^b$	$-\boldsymbol{e}^b\nabla_b\mathcal{V}^a$	0	0
$\frac{\delta}{\delta t}\big\vert^{\mathcal{X}}$	0	0	0	0	0	0	0	0	0	0
$\frac{\delta}{\delta t}$	0	0	0	0	0	0	0	0	0	0

TABLE A2. A table of derivatives of the various components of the Levi-Civita tensor $\boldsymbol{\varepsilon}$. These can be calculated with the help of Grinfeld (2013) equations (9.63)-(9.67) laying out derivatives of the metric determinant g.

Derivative	ε_{ijk}	ε^{ijk}	ε_{abc}	ε^{abc}
∂_ℓ	$\varepsilon_{ijk}\Gamma^m{}_{m\ell}$	$-\varepsilon^{ijk}\Gamma^m{}_{m\ell}$	$\varepsilon_{abc}\Gamma^f{}_{fd}J^d_\ell$	$-\varepsilon^{abc}\Gamma^f{}_{fd}J^d_\ell$
∂_d	$\varepsilon_{ijk}\Gamma^m{}_{m\ell}J^\ell_d$	$-\varepsilon^{ijk}\Gamma^m{}_{m\ell}J^\ell_d$	$\varepsilon_{abc}\Gamma^f{}_{fd}$	$-\varepsilon^{abc}\Gamma^f{}_{fd}$
$\frac{\partial}{\partial t}\big\vert_x$	$\varepsilon_{ijk}\nabla_m\mathcal{V}^m$	$-\varepsilon^{ijk}\nabla_m\mathcal{V}^m$	$\varepsilon_{abc}\Gamma^f{}_{fd}\mathcal{V}^d$	$-\varepsilon^{abc}\Gamma^f{}_{fd}\mathcal{V}^d$
$\frac{\partial}{\partial t}\big\vert_y$	$\varepsilon_{ijk}\partial_m\mathcal{V}^m$	$-\varepsilon^{ijk}\partial_m\mathcal{V}^m$	0	0
∇_ℓ	0	0	0	0
∇_d	0	0	0	0
$\mathcal{L}_{\mathcal{V}}$	$\varepsilon_{ijk}\nabla_m\mathcal{V}^m$	$-\varepsilon^{ijk}\nabla_m\mathcal{V}^m$	$\varepsilon_{abc}\nabla_f\mathcal{V}^f$	$-\varepsilon^{abc}\nabla_f\mathcal{V}^f$
$\frac{\delta}{\delta t}\big\vert^{\mathcal{X}}$	0	0	0	0
$\frac{\delta}{\delta t}$	0	0	0	0

so these derivatives coincide if and only if $\boldsymbol{v}$ is a Killing field, which would correspond physically to a rigidly translating and rotating fluid – a restrictive constraint indeed.

We have shown, for the particular case $\boldsymbol{\mathcal{V}} = \boldsymbol{v}$, that the intrinsic derivative and Oldroyd's derivative encode similar information about time-varying tensors, but they are nearly opposite in character. Oldroyd's derivative begins with the components $\partial w^i \partial t|_x$ and asserts that, when transformed as tensor components, these look a certain way in the y^a coordinates. The convected derivative is therefore constructed to produce the second form from the first, and, for vector fields, produces the components of $\partial \boldsymbol{w}/\partial t|_y + [\boldsymbol{v}, \boldsymbol{w}]$. Brackets indicate the commutator operation.

In contrast, the intrinsic derivative begins with the tensor field $d\boldsymbol{w}/dt$, and asks: what are the components of this in each of the bases $\boldsymbol{e}_i$ and $\boldsymbol{e}^i$? The derivative is therefore constructed to act on either set of components w^i or w_i and produce the components of $d\boldsymbol{w}/dt$ in the relevant basis. We prefer this latter conceptual framing because the

TABLE A3. A table of derivative commutators. The commutator is understood to be applied to the relevant components of the tensor in each column.

Commutator	w^i
$\left[\frac{\partial}{\partial t}\big\vert_x, \partial_j\right]$	0
$\left[\frac{\partial}{\partial t}\big\vert_x, \nabla_j\right]$	$w^k \nabla_k \nabla_j \mathcal{V}^i$
$\left[\mathcal{L}_{\mathcal{V}}, \nabla_j\right]$	$w^k \nabla_k \nabla_j \mathcal{V}^i$
$\left[\frac{\delta}{\delta t}\big\vert^{\mathcal{X}}, \nabla_j\right]$	$-(\nabla_j q^k)(\nabla_k w^i)$
$\left[\frac{\delta}{\delta t}, \nabla_j\right]$	$-(\nabla_j v^k)(\nabla_k w^i)$
$\left[\frac{\partial}{\partial t}\big\vert_y, \partial_b\right]$	0
$\left[\frac{\partial}{\partial t}\big\vert_y, \nabla_b\right]$	0
$[\mathcal{L}_{\mathcal{V}}, \nabla_b]$	$w^c \nabla_c \nabla_b \mathcal{V}^a$
$\left[\frac{\delta}{\delta t}\big\vert^{\mathcal{X}}, \nabla_b\right]$	$-(\nabla_b q^c)(\nabla_c w^a)$
$\left[\frac{\delta}{\delta t}, \nabla_b\right]$	$-(\nabla_b v^c)(\nabla_c w^a)$

relevant coordinate-free field $d\boldsymbol{w}/dt$ is easier to interpret than $\partial \boldsymbol{w}/\partial t|_y + [\boldsymbol{v}, \boldsymbol{w}]$.

APPENDIX C

Worked Example: Vertical Expansion

Define the coordinates

$$Y^a = \begin{pmatrix} x \\ y \\ z(\zeta, t) \end{pmatrix} \qquad X^i = \begin{pmatrix} x \\ y \\ \zeta(z, t) \end{pmatrix} \tag{C1}$$

giving us Jacobians

$$J^a_i \equiv \frac{\partial Y^a}{\partial X^i} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\partial z}{\partial \zeta} \end{bmatrix} \qquad J^i_a \equiv \frac{\partial X^i}{\partial Y^a} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\partial \zeta}{\partial z} \end{bmatrix} \tag{C2}$$

and setting velocity

$$\mathcal{V}^a \equiv \left.\frac{\partial Y^a}{\partial t}\right|_X = \begin{bmatrix} 0 \\ 0 \\ \frac{\partial z}{\partial t}\big\vert_\zeta \end{bmatrix} \qquad \mathcal{V}^i = J^i_a \mathcal{V}^a = \begin{bmatrix} 0 \\ 0 \\ \frac{\partial \zeta}{\partial z} \frac{\partial z}{\partial t}\big\vert_\zeta \end{bmatrix}$$

$$\mathcal{V}_a = g_{ab} \mathcal{V}^b = \begin{bmatrix} 0 \\ 0 \\ \frac{\partial z}{\partial t}\big\vert_\zeta \end{bmatrix} \qquad \mathcal{V}_i = \mathcal{V}_a J^a_i = \begin{bmatrix} 0 \\ 0 \\ \frac{\partial z}{\partial \zeta} \frac{\partial z}{\partial t}\big\vert_\zeta \end{bmatrix} \tag{C3}$$

where $g_{ab} = \delta_{ab}$ because the Y frame is Cartesian. This also lets us simplify the gradient tensor

$$\nabla_b \mathcal{V}_a = \partial_b \mathcal{V}_a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \frac{\partial z}{\partial t}\big\vert_\zeta \end{bmatrix} \tag{C4}$$

where a indexes rows and b columns. Then

$$\begin{aligned} \nabla_j \mathcal{V}_i &= J^b_j J^a_i \nabla_b \mathcal{V}_a \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\partial z}{\partial \zeta} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \frac{\partial z}{\partial t}\big\vert_\zeta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\partial z}{\partial \zeta} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \left(\frac{\partial z}{\partial \zeta}\right)^2 \frac{\partial}{\partial z} \frac{\partial z}{\partial t}\big\vert_\zeta \end{bmatrix} \end{aligned} \tag{C5}$$

so

$$\nabla_{(j} \mathcal{V}_{i)} = \nabla_j \mathcal{V}_i \qquad \nabla_{[j} \mathcal{V}_{i]} = 0 \tag{C6}$$

and the Coriolis and centrifugal-like forces are identically zero. The fantastical force is

$$\begin{aligned} \mathcal{V}^j \nabla_{(j} \mathcal{V}_{i)} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \left(\frac{\partial z}{\partial \zeta}\right)^2 \frac{\partial}{\partial z} \frac{\partial z}{\partial t}\big\vert_\zeta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \frac{\partial \zeta}{\partial z} \frac{\partial z}{\partial t}\big\vert_\zeta \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ \frac{\partial z}{\partial \zeta} \frac{\partial}{\partial z}\left(\frac{\partial z}{\partial t}\big\vert_\zeta\right) \frac{\partial z}{\partial t}\big\vert_\zeta \end{bmatrix} \end{aligned} \tag{C7}$$

and the setting tendency force is

$$-\left.\frac{\partial \mathcal{V}_i}{\partial t}\right|_X = \begin{bmatrix} 0 \\ 0 \\ -\frac{\partial}{\partial t}\big\vert_X \left(\frac{\partial z}{\partial \zeta} \frac{\partial z}{\partial t}\big\vert_\zeta\right) \end{bmatrix}. \tag{C8}$$

a. A simple, non-trivial choice

Choose

$$\zeta(z,t) = z(1+\frac{t}{T}) \qquad z = \frac{\zeta}{1+\frac{t}{T}} \tag{C9}$$

for some timescale T so that $\zeta = z$ when $t = 0$. Then

$$\frac{\partial z}{\partial \zeta} = \frac{1}{1+\frac{t}{T}} \qquad \frac{\partial \zeta}{\partial z} = 1+\frac{t}{T} \tag{C10}$$

$$\left.\frac{\partial z}{\partial t}\right|_{\zeta} = \frac{-\zeta}{T(1+\frac{t}{T})^2} = \frac{-z}{T(1+\frac{t}{T})} \tag{C11}$$

$$J^a_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/(1+\frac{t}{T}) \end{bmatrix} \qquad J^i_a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+\frac{t}{T} \end{bmatrix} \tag{C12}$$

$$\mathcal{V}^a = \begin{bmatrix} 0 \\ 0 \\ \frac{-z}{T(1+\frac{t}{T})} \end{bmatrix} \qquad \mathcal{V}_a = \begin{bmatrix} 0 \\ 0 \\ \frac{-z}{T(1+\frac{t}{T})} \end{bmatrix} \tag{C13}$$

$$\mathcal{V}^i = \begin{bmatrix} 0 \\ 0 \\ -\frac{z}{T} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{-\zeta}{T(1+\frac{t}{T})} \end{bmatrix} \qquad \mathcal{V}_i = \begin{bmatrix} 0 \\ 0 \\ \frac{-z}{T(1+\frac{t}{T})^2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{-\zeta}{T(1+\frac{t}{T})^3} \end{bmatrix} \tag{C14}$$

$$\nabla_b \mathcal{V}_a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{T(1+\frac{t}{T})} \end{bmatrix} \qquad \nabla^b \mathcal{V}^a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{T(1+\frac{t}{T})} \end{bmatrix} \tag{C15}$$

$$\nabla_j \mathcal{V}_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{T(1+\frac{t}{T})^3} \end{bmatrix} \qquad \nabla^j \mathcal{V}^i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{(1+\frac{t}{T})}{T} \end{bmatrix} \tag{C16}$$

Because the gradient tensor is symmetric, there are no Coriolis-like or centrifugal-like forces, and $\mathcal{V}^j \nabla_{(j} \mathcal{V}_{i)} = -\nabla_i \Phi$ for the setting potential:

$$\Phi = -\frac{1}{2}\mathcal{V}_i \mathcal{V}^i = -\frac{1}{2}\left(\frac{\zeta}{T(1+\frac{t}{T})^2}\right)^2. \tag{C17}$$

The conservative part of the apparent force is then

$$-\nabla_i \Phi = \begin{bmatrix} 0 \\ 0 \\ \frac{\zeta}{T^2(1+\frac{t}{T})^4} \end{bmatrix}, \tag{C18}$$

the setting tendency force is

$$-\left.\frac{\partial \mathcal{V}_i}{\partial t}\right|_X = \begin{bmatrix} 0 \\ 0 \\ \frac{-3\zeta}{T^2(1+\frac{t}{T})^4} \end{bmatrix}, \tag{C19}$$

and the total apparent force for covariant components is

$$-\left.\frac{\partial \boldsymbol{\mathcal{V}}}{\partial t}\right|_X \cdot \boldsymbol{e}_i = -\nabla_i \Phi - \left.\frac{\partial \mathcal{V}_i}{\partial t}\right|_X = \begin{bmatrix} 0 \\ 0 \\ \frac{-2\zeta}{T^2(1+\frac{t}{T})^4} \end{bmatrix}. \tag{C20}$$

In contravariant components, the apparent force will have a piece dependent on the fluid velocity, which we define to be

$$v_a = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \qquad v_i = \begin{bmatrix} v_x \\ v_y \\ \frac{v_z}{1+\frac{t}{T}} \end{bmatrix} \tag{C21}$$

Recognizing that $2\boldsymbol{u} + \boldsymbol{\mathcal{V}} = 2\boldsymbol{v} - \boldsymbol{\mathcal{V}}$, the total apparent force is

$$\begin{aligned} &-(2v_j - \mathcal{V}_j)\nabla^{(j}\mathcal{V}^{i)} - \left.\frac{\partial \mathcal{V}^i}{\partial t}\right|_X \qquad \text{(C22)} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{(1+\frac{t}{T})}{T} \end{bmatrix} \left(\begin{bmatrix} 2v_x \\ 2v_y \\ \frac{2v_z}{1+\frac{t}{T}} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{\zeta}{T(1+\frac{t}{T})^3} \end{bmatrix} \right) - \begin{bmatrix} 0 \\ 0 \\ \frac{\zeta}{T^2(1+\frac{t}{T})^2} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ \frac{2v_z}{T} \end{bmatrix} \end{aligned}$$

There is a notable contrast between the character of the apparent forces in this example as they appear in covariant components (C20) and contravariant components (C22): the former depend entirely on the absolute motion $\boldsymbol{\mathcal{V}}$ of the coordinates, while the latter are proportional to the absolute motion $\boldsymbol{v}$ of the fluid.

References

Åhlander, K., M. Haveraaen, and H. Z. Munthe-Kaas, 2001: On the role of mathematical abstractions for scientific computing. *The Architecture of Scientific Software: IFIP TC2/WG2. 5 Working Conference on the Architecture of Scientific Software October 2–4, 2000, Ottawa, Canada*, Springer, 145–158.

Åhlander, K., and K. Otto, 2006: Software design for finite difference schemes based on index notation. *Future Generation Computer Systems*, **22 (1-2)**, 102–109.

Aris, R., 1962: *Vectors, tensors and the basic equations of fluid mechanics*. Dover Publications, Inc., Mineola, NY, 286 pp.

Batchelor, G., 2000: *An introduction to fluid dynamics*. Cambridge University Press, Cambridge, UK, 615 pp., doi:10.1017/CBO9780511800955.

Bleck, R., 2002: An oceanic general circulation model framed in hybrid isopycnic-cartesian coordinates. *Ocean modelling*, **4 (1)**, 55–88.

Carroll, S. M., 2004: *Spacetime and geometry*. Addison Wesley, 513 pp., doi:10.1017/9781108770385.005.

Cauchy, A. L. B., 1816: *Théorie de la propagation des ondes à la surface d'un fluide pesant d'une profondeur indéfinie*. Académie royale des sciences.

Cotter, C., and J. Thuburn, 2014: A finite element exterior calculus framework for the rotating shallow-water equations. *J. Comput. Phys.*, **257**, 1506–1526, doi:10.1016/j.jcp.2013.10.008.

Crane, K., 2018: Discrete differential geometry: An applied introduction. *Notices of the AMS, Communication*, **1153**.

Crane, K., F. de Goes, M. Desbrun, and P. Schröder, 2013: Digital geometry processing with discrete exterior calculus. *ACM SIGGRAPH 2013 Courses*, Association for Computing Machinery, New York, NY, USA, SIGGRAPH '13, doi:10.1145/2504435.2504442, URL https://doi.org/10.1145/2504435.2504442.

Dimitrienko, Y. I., 2011: *Nonlinear continuum mechanics and large inelastic deformations*. Springer, Dordrecht, Netherlands, 721 pp., doi:10.1007/978-94-007-0034-5.

Dukowicz, J. K., and R. D. Smith, 1997: Stochastic theory of compressible turbulent fluid transport. *Physics of Fluids*, **9 (11)**, 3523–3529.

Engø, K., A. Marthinsen, and H. Z. Munthe-Kaas, 2001: Diffman: An object-oriented matlab toolbox for solving differential equations on manifolds. *Applied Numerical Mathematics*, **39 (3)**, 323–347, doi:https://doi.org/10.1016/S0168-9274(00)00042-8, URL https://www.sciencedirect.com/science/article/pii/S0168927400000428, themes in Geometric Integration.

Feske, J., B. Fox-Kemper, and J. Lilly, 2026: Beyond vertical coordinates: A generalized framework for ocean model equations with anisotropic diffusivity. In prep.

Fox-Kemper, B., and Coauthors, 2019: Challenges and prospects in ocean circulation models. *Front. Marine Sci.*, **6**, doi:10.3389/fmars.2019.00065.

Friis, H. A., T. A. Johansen, M. Haveraaen, H. Munthe-Kaas, and Åsmund Drottning, 2001: Use of coordinate-free numerics in elastic wave simulation. *Applied Numerical Mathematics*, **39 (2)**, 151–171, doi:https://doi.org/10.1016/S0168-9274(01)00096-4, URL https://www.sciencedirect.com/science/article/pii/S0168927401000964.

Frisch, U., and B. Villone, 2014: Cauchy's almost forgotten lagrangian formulation of the euler equation for 3d incompressible flow. *The European Physical Journal H*, **39 (3)**, 325–351, doi:10.1140/epjh/e2014-50016-6, URL https://doi.org/10.1140/epjh/e2014-50016-6.

Gibson, A. H., A. M. Hogg, A. E. Kiss, C. J. Shakespeare, and A. Adcroft, 2017: Attribution of horizontal and vertical contributions to spurious mixing in an arbitrary lagrangian–eulerian ocean model. *Ocean Modelling*, **119**, 45–56.

Gilbert, A. D., and J. Vanneste, 2023: A geometric look at momentum flux and stress in fluid mechanics. *Journal of Nonlinear Science*, **33 (2)**, 31, doi:10.1007/s00332-023-09887-0.

Gill, A. E., 1982: *Atmosphere–ocean dynamics*, International Geophysics Series, Vol. 30. Academic Press, San Diego, CA, 662 pp., URL https://archive.org/details/AtmosphereOceanDynamicsGill.

Grant, P. W., M. Haveraaen, and M. F. Webster, 2000: Coordinate free programming of computational fluid dynamics problems. *Scientific Programming*, **8 (4)**, 419 840, doi:https://doi.org/10.1155/2000/419840, URL https://onlinelibrary.wiley.com/doi/abs/10.1155/2000/419840, https://onlinelibrary.wiley.com/doi/pdf/10.1155/2000/419840.

Griffies, S. M., 1998: The gent–mcwilliams skew flux. *Journal of Physical Oceanography*, **28 (5)**, 831–841.

Griffies, S. M., 2004: *Fundamentals of ocean climate models*. Princeton University Press, doi:10.1515/9780691187129.

Griffies, S. M., A. Adcroft, and R. W. Hallberg, 2020: A primer on the vertical lagrangian-remap method in ocean models based on finite volume generalized vertical coordinates. *Journal of Advances in Modeling Earth Systems*, **12 (10)**, e2019MS001 954.

Griffies, S. M., R. C. Pacanowski, and R. W. Hallberg, 2000: Spurious diapycnal mixing associated with advection in a z-coordinate ocean model. *Monthly Weather Review*, **128 (3)**, 538–564.

Griffies, S. M., and Coauthors, 2015: Impacts on ocean heat from transient mesoscale eddies in a hierarchy of climate models. *J. Climate*, **28 (3)**, 952–977, doi:10.1175/jcli-d-14-00353.1.

Grinfeld, P., 2009: Exact nonlinear equations for fluid films and proper adaptations of conservation theorems from classical hydrodynamics. *Journal of Geometry and Symmetry in Physics*, **16**, 1–21.

Grinfeld, P., 2013: *Introduction to tensor analysis and the calculus of moving surfaces*. Springer Science+Business Media, New York, NY, 302 pp., doi:10.1007/978-1-4614-7867-6.

Halliwell, G. R., 2004: Evaluation of vertical coordinate and vertical mixing algorithms in the hybrid-coordinate ocean model (hycom). *Ocean Modelling*, **7 (3-4)**, 285–322.

Haveraaen, M., and H. A. Friis, 2009: Coordinate-free numerics: all your variation points for free? *International Journal of Computational Science and Engineering*, **4 (4)**, 223–230, doi:10.1504/IJCSE.2009.029159, URL https://www.inderscienceonline.com/doi/abs/10.1504/IJCSE.2009.029159, https://www.inderscienceonline.com/doi/pdf/10.1504/IJCSE.2009.029159.

Haveraaen, M., H. A. Friis, and H. Munthe-Kaas, 2005: Computable scalar fields: A basis for pde software. *The Journal of Logic and Algebraic Programming*, **65 (1)**, 36–49, doi:https://doi.org/10.1016/j.jlap.2004.12.001, URL https://www.sciencedirect.com/science/article/pii/S1567832604001080.

Hawley, J. F., L. L. Smarr, and J. R. Wilson, 1984: A numerical study of nonspherical black hole accretion. i equations and test problems. *The Astrophysical Journal*, **277**, 296–311.

Heinzl, R., and P. Schwaha, 2011: A generic topology library. *Science of Computer Programming*, **76 (4)**, 324–346, doi:https://doi.org/10.1016/j.scico.2009.09.007, URL https://www.sciencedirect.com/science/article/pii/S0167642309001257, special issue on library-centric software design (LCSD 2006).

Hinch, J., and O. Harlen, 2021: Oldroyd B, and not A? *J. Non-Newton. Fluid*, **298**, 104 668:1–7, doi:10.1016/j.jnnfm.2021.104668.

Holm, D. D., 2015: Variational principles for stochastic fluid dynamics. *P. Roy. Soc. Lond. A Mat.*, **471 (2176)**, 20140 963:1–19, doi:10.1098/rspa.2014.0963.

Holm, D. D., 2025: *Geometric Mechanics-Part III: Broken Symmetry And Composition Of Maps*. World Scientific.

Holm, D. D., J. E. Marsden, and T. S. Ratiu, 1998: The euler–poincaré equations and semidirect products with applications to continuum theories. *Advances in Mathematics*, **137 (1)**, 1–81, doi:https://doi.org/10.1006/aima.1998.1721, URL https://www.sciencedirect.com/science/article/pii/S0001870898917212.

Ilıcak, M., A. J. Adcroft, S. M. Griffies, and R. W. Hallberg, 2012: Spurious dianeutral mixing and the role of momentum closure. *Ocean Modelling*, **45**, 37–58.

Itskov, M., 2015: *Tensor algebra and tensor analysis for engineers*. 4th ed., Springer International Publishing, Cham, Switzerland, 290 pp., doi:10.1007/978-3-319-16342-0.

Jeevanjee, N., 2011: *An introduction to tensors and group theory for physicists*. Springer Science+Business Media, New York, NY, 242 pp., doi:10.1007/978-0-8176-4715-5_1.

Koks, D., 2017: On the use of vectors, reference frames, and coordinate systems in aerospace analysis. Tech. Rep. DST-Group-TR-3309, Defence Science and Technology Group, Australian Department of Defense. URL https://www.dst.defence.gov.au/publication/use-vectors-reference-frames-and-coordinate-systems-aerospace-analysis.

Kundu, P. K., I. M. Cohen, and D. R. Dowling, 2016: *Fluid mechanics*. Sixth ed., Elsevier Inc., doi:10.1016/C2012-0-00611-4.

Lamb, H., 1895: *Hydrodynamics*. Cambridge University Press.

Lilly, J. M., J. Feske, B. Fox-Kemper, and J. J. Early, 2024: Integral theorems for thegradient of a vector field, with a fluid dynamical application. *P. Roy. Soc. Lond. A Mat.*, **480 (2293)**, 20230 550:1–30, doi:10.1098/rspa.2023.0550.

Luo, H., and T. R. Bewley, 2004: On the contravariant form of the navier–stokes equations in time-dependent curvilinear coordinate systems. *Journal of Computational Physics*, **199 (1)**, 355–375, doi:https://doi.org/10.1016/j.jcp.2004.02.012, URL https://www.sciencedirect.com/science/article/pii/S0021999104000889.

Maddison, J. R., and D. Marshall, 2013: The Eliassen-Palm flux tensor. *J. Fluid Mech.*, **729**, 69–102, doi:10.1017/jfm.2013.259.

Marsden, J. E., and T. S. Ratiu, 1999: *Introduction to mechanics and symmetry*. 2nd ed., Texts in Applied Mathematics, Springer, New York, NY, 584 pp., doi:10.1007/978-0-387-21792-5.

McDougall, T. J., 1987: Neutral surfaces. *Journal of Physical Oceanography*, **17 (11)**, 1950–1964.

Megann, A., J. Chanut, and D. Storkey, 2022: Assessment of the z time-filtered arbitrary lagrangian-eulerian coordinate in a global eddy-permitting ocean model. *Journal of Advances in Modeling Earth Systems*, **14 (11)**, e2022MS003 056.

Morin, D., 2008: *Introduction to classical mechanics*. Cambridge University Press, Cambridge, UK, 719 pp., doi:10.1017/cbo9780511808951.

Needham, T., 2021: *Visual differential geometry and forms*. Princeton University Press, Princeton, NJ, 501 pp., doi:10.1515/9780691219899-004.

Ogden, R. W., 1984: *Non-linear elastic deformations*. Ellis Horwood Limited, Mineola, NY, 532 pp., URL https://archive.org/details/nonlinearelastic0000ogde.

Oldroyd, J. G., 1950: On the formulation of rheological equations of state. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, **200 (1063)**, 523–541.

Pedlosky, J., 1987: *Geophysical fluid dynamics*. 2nd ed., Springer-Verlag, New York, NY, 710 pp., doi:10.1007/978-1-4612-4650-3.

Petersen, M. R., D. W. Jacobsen, T. D. Ringler, M. W. Hecht, and M. E. Maltrud, 2015: Evaluation of the arbitrary lagrangian–eulerian vertical coordinate method in the mpas-ocean model. *Ocean Modelling*, **86**, 93–113.

Prusa, J. M., 2018: Computation at a coordinate singularity. *Journal of Computational Physics*, **361**, 331–352.

Salmon, R., 2013: An alternative view of generalized Lagrangian mean theory. *J. Fluid Mech.*, **719**, 165–182, doi:10.1017/jfm.2012.638.

Thiffeault, J.-L., 2001: Covariant time derivatives for dynamical systems. *Journal of Physics A: Mathematical and General*, **34 (29)**, 5875–5885, doi:10.1088/0305-4470/34/29/309.

Thorne, K. S., and R. D. Blandford, 2017: *Modern classical physics: optics, fluids, plasmas, elasticity, relativity, and statistical physics*. Princeton University Press.

Vallis, G. K., 2006: *Atmospheric and oceanic fluid dynamics: fundamentals and large-scale circulation*. Cambridge University Press, Cambridge, UK, 946 pp., doi:10.1017/9781107588417.

Vallis, G. K., 2017: *Atmospheric and oceanic fluid dynamics*. Cambridge University Press.

Veronis, G., 1975: The role of models in tracer studies. *Numerical Models of the Ocean Circulation: Proceedings of a Symposium Held at Durham, New Hampshire, October 17–20, 1972*, National Academy of Sciences, 133–146.

Young, W. R., 2012: An exact thickness-weighted average formulation of the Boussinesq equations. *J. Phys. Oceanogr.*, **42 (5)**, 692–707, doi:10.1175/JPO-D-11-0102.1.