

Nonequilibrium oscillations, probability angular momentum, and the climate system

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Submitted March 17, 2019

Abstract Though the Boltzmann-Gibbs framework of equilibrium statistical mechanics has been successful in many arenas, it is clearly inadequate for describing many interesting natural phenomena driven far from equilibrium. The simplest step towards that goal is a better understanding of nonequilibrium *steady-states* (NESS). Here we focus on one of the distinctive features of NESS – persistent probability currents – and their manifestations in our climate system. We consider the natural variability of the steady-state climate system, which can be approximated as a NESS. These currents must form closed loops, which are odd under time reversal, providing the crucial difference between systems in thermal equilibrium and NESS. Seeking

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manifestations of such current loops leads us naturally to the notion of “probability angular momentum” and oscillations in the space of observables. Specifically, we will relate this concept to the *asymmetric* part of certain time-dependent correlation functions. Applying this approach, we propose that these current loops give rise to preferred spatio-temporal patterns of natural climate variability that take the form of climate oscillations such as the El-Niño Southern Oscillation (ENSO) and the Madden-Julien Oscillation (MJO). In the space of climate indices, we observe persistent currents and define a new diagnostic for these currents: the probability angular momentum (\mathcal{L}). Using the observed climatic time series of ENSO and MJO, we compute both the averages and the distributions of \mathcal{L} . These results are in good agreement with the analysis from a linear Gaussian model. We propose that, in addition to being a new quantification of climate oscillations across models and observations, the probability angular momentum provides a meaningful characterization for all statistical systems in NESS.

Keywords nonequilibrium steady state · probability currents · climate · El-Niño · Madden-Julien oscillation

1 Introduction

Nearly all the interesting phenomena around us emerge from tractable interactions between simple constituents, e.g., electromagnetism and atoms. However, understanding how emergent phenomena arise - the goal of statistical mechanics - is extremely challenging. For systems in thermal equilibrium, Boltzmann and Gibbs provided a highly successful framework, while linear response theory is adequate for describing systems *near* equilibrium, see e.g., [37]. Yet, most fascinating phenomena in nature are associated with systems driven far from equilibrium [7], e.g., all life forms, socio-political structures, and the climate system. In particular, such systems would either not exist or be vastly different under conditions of thermal equilibrium, i.e., when they are totally isolated or allowed to exchange energy (or particles, or information [32]) with just one reservoir. Despite much progress on fluctuation theorems and the “nonequilibrium counterpart” of the free energy in recent years (see, e.g., Ref. [40]), an overarching framework for far-from-equilibrium systems remains elusive. Often, to study such interesting systems, we rely on models with a few (macroscopic) degrees of freedom, evolving as nonequilibrium stochastic processes. One frequently used approach involves master or Fokker-Planck equations for the probability distribution, with time-independent rates. While analyzing the full time dependence is generally beyond our reach, we can take initial steps, by studying the associated stationary states (which are guaranteed to exist). If these rates obey detailed balance, the stationary distribution can be easily found and the system can be treated as if it is in thermal equilibrium [37]. On the other hand, if the rates *violate* detailed balance, then even finding the stationary distribution is highly non-trivial in general [13]. Specifically, such detailed balance-violating systems settle into nonequilibrium steady-states (NESS), and understanding their properties (e.g., fluctuations and correlations) is quite challenging. In particular, unlike systems in thermal equilibrium,

there are persistent probability currents [53], which form closed loops and characterize underlying rotations in configuration space. Studying the observable consequences of such steady current loops is surely a valuable endeavor, and is likely to lead to fruitful insights for all NESS. Here we focus on one such observable - the probability angular momentum, in analogy with the familiar angular momenta associated with fluid current loops (e.g., [43]). As shown below, this quantity is intimately related to fluctuations and temporal correlations in the NESS. Introduced recently in other contexts [41, 28, 29, 54], it will be considered here in the context of the Earth's climate system.

The climate system is forced by incoming short-wave solar radiation and it is damped by outgoing infrared radiation emitted to space, with a distribution of net radiative forcing segregated by latitude. As a result, the climate system is approximately in a NESS [23, 34, 33, 24, 25, 21]. While this approximation is violated by non-steady forcings such as solar variability, seasons and Milankovitch cycles in the Earth's orbit, intermittent volcanic eruptions, and anthropogenic greenhouse forcing, much remains to be learned about the steady-state climate. In this study, we will ignore non-steady forcings. The climate system is known to exhibit many self-organized, irregular, spatio-temporal patterns, typically referred to as oscillations. These patterns include the El-Niño Southern Oscillation (ENSO) [46, 47, 27], the Madden-Julien Oscillation (MJO) [18], the Pacific Decadal Oscillation (PDO) [31], and the Atlantic Multidecadal Oscillation (AMO) [52]. It should be emphasized that these "oscillations" are not single-frequency constant amplitude sinusoidal fluctuations or necessarily wavelike phenomena: their frequency distributions and amplitude variations are broad and important, but they are narrow enough that each one is an empirically recognized coherent spatio-temporal pattern of natural variability. ENSO and the MJO are emergent phenomena that result from a complex "organization" of dynamical processes including tropical convection, the velocities and temperatures of the atmosphere and ocean, and large-scale oceanic and atmospheric waves [30, 16]. As such, they are unlikely to obey detailed balance or any particular time-reversal symmetry. Nonetheless, ENSO and the MJO are the dominant modes of equatorial interannual and tropical intraseasonal variability, respectively. So, ENSO and the MJO both emerge from a multiplicity of mechanisms, and dominate other variability in their regions and timescales. These climate oscillations are seen as fluctuations about the mean climate state and we interpret the specific spatio-temporal character of the oscillations as the physical-space manifestation of the probability currents in the phase space¹ of the climate system.

Climate oscillations are often characterized by climate "indices". These indices are empirically determined combinations of climate variables, typically chosen by researchers in the subject area of interest to highlight the most important features of

¹ In much of the physics community, "phase space" is a term used for the space of $x-p$ (coordinate and momentum). Significantly, these variables are even/odd under time reversal. In this paper, however, we use this term in the sense common in the dynamical systems and climate science communities. In the cases we consider here, there is no reason to regard the variables (e.g., temperature and volume, or two amplitudes of a principal component analysis) as having different symmetry under time reversal. For many in the community of statistical physics, the familiar term in this context is "configuration space." We will use the two terms interchangeably and assume there is no confusion.

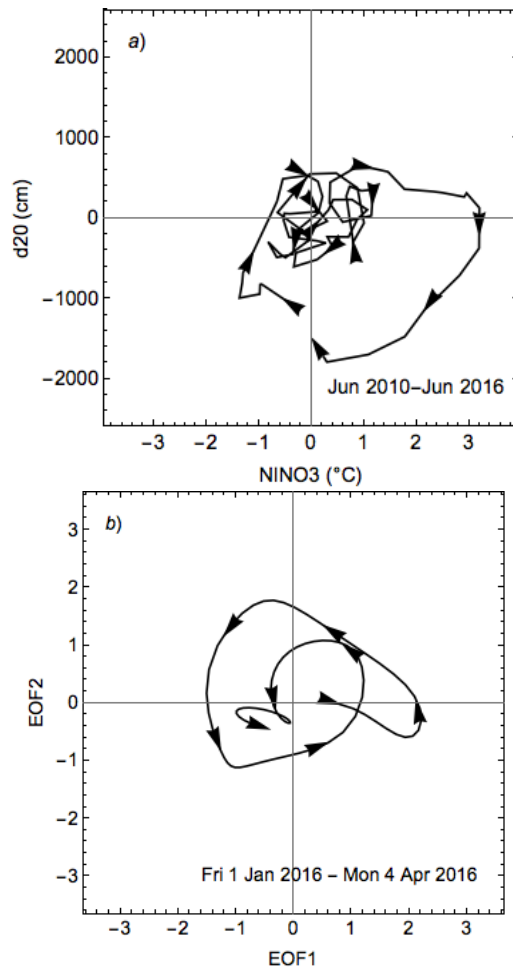


Fig. 1 Phase space trajectories for a) ENSO and b) the MJO

a particular phenomenon. It is not uncommon for different researchers to define different indices to highlight different aspects of the same complex high-dimensional phenomenon. While it is most common to focus on a single index, sometimes two indices are used to describe an oscillation and investigate the trajectories of the indices in the resulting two-dimensional phase space. Climate oscillations are then observed to have trajectories which exhibit phase space rotation. For example, ENSO is often described in terms of the NINO3 index, based on the spatially-averaged Sea Surface Temperature in the eastern tropical Pacific (90°W to 150°W and 5°S to 5°N), and the average depth of the 20°C isotherm over the same area, which is a measure of the volume of warm water in the tropical Pacific. The two-dimensional phase space of these indices clearly shows the rotation characteristic of fluctuations within NESS, as seen in Figure 1a (e.g. [27]). Similar phase space rotation is seen (Figure 1b) in

a multivariate MJO index [49] based on spatial patterns of variability of outgoing long-wave radiation anomalies, which are a convenient observable closely related to changes in cloud cover. We propose a novel and natural measure to quantify such rotations: the probability angular momentum (PAM). Not only is it intimately related to the underlying probability current loops in phase space, it is readily computed from both observations and models, providing researchers in the climate community a new tool for model diagnosis, validation and intercomparison.

One of the simplest classes of models that captures nonequilibrium steady-states are Langevin models based on multivariate linear stochastic differential equations with additive noise. Since their introduction by Uhlenbeck and Orstein [45], these models has been continuously applied to many problems in the physics community. They are also used to model aspects of the climate system, and, when constructed by fitting to data, are known as linear inverse models, e.g. [35]. Generally assumed to have Gaussian noise and referred to as Linear Gaussian Models (LGM), they have been successfully used to describe a variety of climate oscillations [35, 36, 1, 10, 4, 5, 42]. The simplicity of these LGMs allows the properties of the NESS to be calculated analytically and facilitates the presentation of PAM. We emphasize that, although we analyze the PAM in the context of LGMs, the quantity itself is quite general, can be calculated from observations with no assumptions about underlying models, and captures the phase space rotation for any system regardless of the underlying dynamics.

For the convenience of readers in the climate science community, we provide in the next section a brief review of the role played by probability current loops in NESSs in general and in LGMs in particular. Section 3 introduces the probability angular momentum - its average as well as its full distribution. In section 4 we calculate the probability angular momentum of two climate oscillations: ENSO and the MJO. We end with a summary and outlook.

2 Nonequilibrium Steady-states and Probability Currents

The highly successful Boltzmann-Gibbs framework for equilibrium statistical mechanics is based on a single hypothesis, that an *isolated* physical system can be found (starting with any initial state but waiting for a time long compared to all intrinsic relaxation time-scales) in any of its allowed configurations, $\{C\}$, with equal probability, i.e., the probability distribution function (pdf) being² $P_{iso}^*(C) \propto 1$. Considering two systems that can exchange energy (or particles, or other quantities), but otherwise isolated, we arrive at a well defined notion of thermal equilibrium and in particular, the Boltzmann pdf: $P_B^* \propto e^{-\mathcal{H}/k_B T}$, where \mathcal{H} is the energy of a configuration C . Further, the dynamical behavior in the equilibrium state is symmetric under time-reversal: One cannot distinguish (statistically) a movie taken of this system from another run in reverse. However, this framework fails to describe many interesting nonequilibrium phenomena and, in particular, the climate system. One reason is clear: The climate

² Below, we will be considering time-dependent distributions, which we denote by $P(C;t)$. The superscript (*) signifies a stationary distribution.

system is driven by incoming shortwave solar energy, balanced by outgoing long-wave radiation. Roughly speaking, it is coupled to two thermal reservoirs, the Sun at 6000°K and the cosmic microwave background at 3°K . The climate system is best regarded as a (approximate) nonequilibrium steady-state. In general, there is no simple way to find, or to hypothesize, the probability distribution of a NESS. Though such a state is, by definition, invariant under time *translation*, it is not so under time reversal. To make progress for this challenging problem, we may start with a master equation³ for the evolution of the pdf:

$$\frac{\partial}{\partial t}P(C;t) = \sum_{C'} [W(C' \rightarrow C)P(C';t) - W(C \rightarrow C')P(C;t)] \quad (1)$$

by postulating a set of (non-negative) rates, $W(C' \rightarrow C)$, for the system to make a transition from C' to C . As probability is conserved, this equation can be regarded as a continuity equation for the density P , with the terms on the right representing net probability currents, $K(C' \rightarrow C)$, from C' to C . As the system settles into stationarity, $P(C;t) \rightarrow P^*(C)$ (which is unique if the W 's allow the system to reach to all C 's), while K settles into K^* . Of course, the sum of the K^* 's into each C must vanish. The principal difference between systems in thermal equilibrium and NESS is that, in the former, *every* K^* vanish. A set of W 's which leads to such a condition, $W(C' \rightarrow C)P^*(C') = W(C \rightarrow C')P^*(C)$, is said to obey detailed balance (DB)⁴. To model general stochastic processes, we typically encounter W 's which violate DB. Those systems settle into NESS, with some non-trivial and persistent probability currents. Being in the stationary state, these K^* 's must form closed loops. Our goal is to find observable manifestations of such persistent current loops and we will find that these quantities are automatically *odd* under time reversal. Further details of this approach may be found in ref. [53].

For a large variety of physical systems, a more restrictive version of Eqn. (1) is quite adequate, namely, the Fokker-Planck equation [38]. Specifically, suppose our configuration space consists of N real variables – $\mathbf{x} \in \mathbb{R}^N$ or x_α , $\alpha = 1, \dots, N$ – and the only non-vanishing transitions – $W(C' \rightarrow C)$ – take the system from C' to infinitesimally nearby C 's. Then, Eqn. (1) reduces to

$$\frac{\partial}{\partial t}P(\mathbf{x},t) = \sum_{\alpha} \frac{\partial}{\partial x_{\alpha}} \left[\sum_{\beta} \frac{\partial}{\partial x_{\beta}} \{D_{\alpha\beta}(\mathbf{x})P(\mathbf{x},t)\} - \mu_{\alpha}(\mathbf{x})P(\mathbf{x},t) \right] \quad (2)$$

where $D_{\alpha\beta}$ is referred to as the diffusion tensor and μ_{α} , the drift vector. The advantage of this form is the correspondence to the continuity equation in, say, fluid dynamics: $\partial\rho/\partial t = -\nabla \cdot \mathbf{J}$, where $\rho(\mathbf{x},t)$ is the fluid density field and \mathbf{J} , the current (density). In this way, we identify the probability current (density) as

$$K_{\alpha} = \mu_{\alpha}P - \partial_{\beta}D_{\alpha\beta}P \quad (3)$$

³ Though the form of our equation appears to be for continuous t and discrete C , it is simple to write equations for other types of variables, e.g., continuous t and C . Note that we have restricted ourselves to systems evolving with time-*independent* rates.

⁴ In this form, the criterion for the W 's to satisfy DB appears to depend on P^* . Kolmogorov [19] provided a criterion which involves only the W 's.

where repeated indices are summed. Similarly, in analogy with $\mathbf{J} = \rho \mathbf{v}$ in fluids, we identify the velocity field as

$$u_\alpha(\mathbf{x}, t) = \mu_\alpha(\mathbf{x}) - \partial_\beta D_{\alpha\beta}(\mathbf{x}) - D_{\alpha\beta}(\mathbf{x}) \partial_\beta \ln P(\mathbf{x}, t) \quad (4)$$

Using standard techniques, this stochastic process can be recast as the more intuitive Langevin equation

$$\frac{d\mathbf{x}}{dt} = \boldsymbol{\mu} + \boldsymbol{\eta} \quad (5)$$

Here, we recognize $\boldsymbol{\mu}$ as the deterministic part of this equation of motion (and typically depends only on \mathbf{x}), while $\boldsymbol{\eta}$ is an additive Gaussian noise with zero mean (i.e., $\langle \boldsymbol{\eta} \rangle \equiv 0$) and covariance $\langle \eta_\alpha(t) \eta_\beta(t') \rangle = 2D_{\alpha\beta} \delta(t - t')$. To be specific, we can focus on the Ito formulation of this stochastic differential equation, e.g., discretizing time into steps of ε and letting⁵ $\mathbf{x}(t + \varepsilon) = \mathbf{x}(t) + \varepsilon [\boldsymbol{\mu}(\mathbf{x}(t)) + \boldsymbol{\eta}(t)]$

In the remainder of this section, we will present these ideas in the context of LGMs. Being the simplest version of (5) and exactly solvable [22, 48, 53], they offer a clear and concise setting for us to introduce the notion probability angular momentum. Further, since LGMs are frequently used in the climate community [35, 36, 1, 10, 4, 5], we will exploit them in describing two examples from the global climate system, showcasing characteristics of NESS which cannot fit within the framework of thermal equilibrium.

The LGM is completely specified by two matrices, \mathbb{A} and \mathbb{D} , with *constant* elements $A_{\alpha\beta}$ and $D_{\alpha\beta}$. The former characterizes the deterministic relaxation into the stationary state, while stability of the system requires the real parts of its eigenvalues to be negative. It enters Eqn. (5) through

$$\boldsymbol{\mu} = \mathbb{A}\mathbf{x} \quad (6)$$

and leads to the term ‘‘linear’’ in LGM. The latter, also known as the diffusion matrix, describes the covariance of the noise⁶, and so, must be positive definite. An alternative expression for this noise is that its distribution is Gaussian: $\propto \exp\{-\eta_\alpha \Gamma^{\alpha\beta} \eta_\beta / 2\}$, where $\Gamma^{\alpha\beta}$ is the matrix inverse of $2D_{\alpha\beta} / \varepsilon$. Denoting the pdf for our LGM⁷ by $p(\mathbf{x}, t)$ and the probability current density by $\mathbf{j}(\mathbf{x}, t)$, the Fokker-Planck equation becomes $\partial p / \partial t + \nabla \cdot \mathbf{j} = 0$, with

$$\mathbf{j} = \mathbb{A}\mathbf{x}p - \mathbb{D}\nabla p \quad (7)$$

In analogy with $\mathbf{J} = \rho \mathbf{v}$ in fluid dynamics, we may identify the velocity field as $\mathbf{u} = \mathbb{A}\mathbf{x} - \mathbb{D}\nabla \ln p$. In the stationary state, the pdf, $p^*(\mathbf{x})$, is a Gaussian [22]

$$p^*(\mathbf{x}) = e^{-\mathbf{x}^T \mathbb{C}_0^{-1} \mathbf{x} / 2} / \sqrt{2\pi \det \mathbb{C}_0}. \quad (8)$$

⁵ Note that the discrete version of the δ in the noise correlation is a Kronecker delta of the time steps divided by ε . Note also that there is no correlation between $\mathbf{x}(t)$ and $\boldsymbol{\eta}(t)$, so that $\langle \mathbf{x}(t) \boldsymbol{\eta}(t) \rangle \equiv 0$.

⁶ The $D_{\alpha\beta}$ here is the same as the one introduced above, the only difference being it is restricted to be x -independent in a LGM.

⁷ To avoid confusion, we use different notation for quantities in a LGM from the general case, e.g., p and \mathbf{j} instead of P and \mathbf{K} .

where⁸ $\mathbb{C}_0 = \langle \mathbf{x}\mathbf{x}^T \rangle^*$ is the covariance matrix, related to \mathbb{A} and \mathbb{D} by the generalized fluctuation dissipation relation [22,48] (Einstein relation):

$$\mathbb{A}\mathbb{C}_0 + \mathbb{C}_0\mathbb{A}^T + 2\mathbb{D} = 0. \quad (9)$$

As presented, the LGM is able to describe systems which settle into either thermal equilibrium or a NESS. If the matrices \mathbb{A} and \mathbb{D} satisfy a further constraint, i.e., $\mathbb{A}^{-1}\mathbb{D}$ being symmetric, then DB is satisfied and the system settles into thermal equilibrium with $\mathbb{C}_0 = \mathbb{A}^{-1}\mathbb{D}$ and $\mathbf{j}^* \equiv 0$. Otherwise, $\mathbf{j}^* \neq 0$ and we have a NESS. It is straightforward to compute \mathbf{j}^* which can be written as $-\mathbb{A}\mathbb{C}_0 + \mathbb{D} \nabla p^*$. Further, it is instructive to exploit (9) and write

$$\mathbf{j}^* = \frac{\mathbb{A}}{2} \nabla p^* \quad (10)$$

where

$$\mathbb{A} \equiv \mathbb{C}_0\mathbb{A}^T - \mathbb{A}\mathbb{C}_0 \quad (11)$$

Since \mathbb{A} is manifestly antisymmetric, $\nabla \cdot \mathbf{j}^* = 0$ follows readily. Being divergence free, these currents must form closed loops, leading to the notion of rotations and “angular momenta” - the main focus of the next section.

We end this section with a few remarks. The expression (10) allows us to visualize the \mathbf{j}^* field easily, as p^* can be regarded as a “hill with ellipsoidal contours” ($N - 1$ dimensional sheets) to which ∇p^* is perpendicular. Thus, \mathbb{A} will generate a vector field lying within these contours. If we focus on a two-dimensional ($N = 2$) space, then the contours are ellipses and \mathbf{j}^* is tangent to them. Explicit examples of such \mathbf{j}^* fields can be found in, e.g., various figures in Ref. [28,29]. Using $\mathbf{u} = \mathbf{K}/P$, we may associate \mathbf{j}^*/p^* with a “probability velocity field” (in the NESS) which “carries probability from one configuration (\mathbf{x}) to a nearby one in preferred directions.” For the LGM, it is $\Omega\mathbf{x}$, where, $\Omega = \mathbb{A} + \mathbb{D}\mathbb{C}_0^{-1}$ is an angular velocity, providing us with the frequency of rotation in configuration space. Finally, note that we can decompose $\mathbb{A}\mathbb{C}_0$ into symmetric and antisymmetric parts

$$-\mathbb{A}\mathbb{C}_0 = \mathbb{D} + \frac{\mathbb{A}}{2} \quad (12)$$

Below, we will discuss the significance of this decomposition.

3 Probability Angular Momentum, its Generalization and Distribution

Probability current loops appear to be abstract concepts; how are they manifested in physical observables? Given that we expect rotations, angles and angular velocities (in configuration space) come naturally to mind [39]. However, there are disadvantages to these quantities, such as dependence on the choice of origin for \mathbf{x} and singular properties when the trajectory $\mathbf{x}(t)$ come close to this origin. We will argue that the analog of angular momentum is a better choice, related to many familiar quantities which are normally used to characterize a time series of observables.

⁸ Here, $\langle \theta \rangle^*$ refers to the average in the stationary state: $\int \theta(\mathbf{x}) p^*(\mathbf{x}) d\mathbf{x}$.

In classical mechanics, the angular momentum associated with a point particle of mass m at $\mathbf{r}(t)$ moving with velocity $\mathbf{v}(t)$ is $\mathbf{L}(t) = \mathbf{r} \times m\mathbf{v}$. Clearly, for a collection of such particles, mass m_i at \mathbf{r}_i moving with velocity \mathbf{v}_i , the total angular momentum is just $\sum_i \mathbf{r}_i \times \mathbf{v}_i m_i$. The next step is to generalize to a continuous distribution described by a mass density $\rho(\mathbf{r}, t)$ and a velocity field $\mathbf{v}(\mathbf{r}, t)$, such as a fluid from an Eulerian perspective. From these, we construct $\mathbf{r} \times \mathbf{v} \rho d\mathbf{r}$ and regard $\mathbf{L}(\mathbf{r}, t) = \mathbf{r} \times \mathbf{v}(\mathbf{r}, t) \rho(\mathbf{r}, t)$ as the angular momentum *density*. The *total* angular momentum of the entire distribution is⁹ $\mathbf{L}(t) = \int d\mathbf{r} \rho(\mathbf{r}, t) \mathbf{r} \times \mathbf{v}(\mathbf{r}, t)$, or substituting \mathbf{J} for $\rho\mathbf{v}$,

$$\mathbf{L}(t) = \int d\mathbf{r} \mathbf{r} \times \mathbf{J}(\mathbf{r}, t) \quad (13)$$

From here, we propose the *probability angular momentum* in configuration space as a straightforward analogue, by letting $\mathbf{r} \rightarrow \mathbf{x}$, $\mathbf{v} \rightarrow \mathbf{u}$, $\rho \rightarrow P$, and $\mathbf{J} \rightarrow \mathbf{K}$. Now, in N dimensions, the PAM is no longer a (pseudo-)vector but a (pseudo-)tensor, \mathbb{L} , with components

$$L_{\alpha\beta}(t) = \int d\mathbf{x} (x_\alpha K_\beta(\mathbf{x}, t) - x_\beta K_\alpha(\mathbf{x}, t)) \quad (14)$$

The integrand here can be regarded as the *probability angular momentum density*

$$L_{\alpha\beta}(\mathbf{x}, t) = x_\alpha K_\beta(\mathbf{x}, t) - x_\beta K_\alpha(\mathbf{x}, t) \quad (15)$$

Since P is normalized to unity, the analogue of total mass is simply unity. Thus, if the units of all components of \mathbf{x} are the same, $[x]$, then the units of the PAM (14) are $[x]^2 [t]^{-1}$ – the same as diffusion (cf. below). In a NESS, we have \mathbb{L}^* , i.e., $L_{\alpha\beta}^* = \int d\mathbf{x} (x_\alpha K_\beta^* - x_\beta K_\alpha^*)$ and note one of the salient features of using angular momenta instead of angles: Since $\nabla \cdot \mathbf{K}^* = 0$, \mathbb{L}^* is *independent* of the choice of the origin for \mathbf{x} .

Turning to the LGM (5,6) in the steady state, the elements of \mathbb{L}^* are

$$L_{\alpha\beta}^* = \int d\mathbf{x} (x_\alpha j_\beta^* - x_\beta j_\alpha^*)$$

Substituting (10) and integrating by parts, we find a very simple result for the LGM:

$$\mathbb{L}^* = \mathbb{A} \quad (16)$$

Of course, since $\mathbf{j} \propto p$, with components of \mathbf{x} as coefficients, \mathbb{L}^* is just linear combinations of the two-point correlation \mathbb{C}_0 . Indeed, it is just twice the antisymmetric part of $-\mathbb{A}\mathbb{C}_0$, while Eqn. (12) puts the PAM on the *same footing* as diffusion. It is not an accident that, as noted above, the units of the PAM are those of diffusion. While we are yet to understand the deeper significance underlying angular momenta and diffusion being part of one quantity (associated with the space of probability density), we can regard this relationship as another salient feature of using the PAM instead of angles to characterize rotation in statistical mechanics.

⁹ Note that we use the same letter for both the density of a quantity and the total, the former having an additional argument, \mathbf{x} .

Before continuing onto generalizations and distributions of the PAM, we present a brief summary of an explicitly analyzed, simple LGM [53,54] which may provide a helpful setting for more complex and realistic phenomena. Consider a system with just two degrees of freedom $(x_{1,2})$, specifically, two coupled simple harmonic oscillators governed by a Hamiltonian, $\mathcal{H} = [k_1 x_1^2 + k_2 x_2^2 + k_\times (x_1 - x_2)^2] / 2$, each immersed in its own thermal bath of temperature $T_{1,2}$. In the low-mass overdamped limit, this system is naturally described by Langevin equations (Boltzmann's constant absorbed into T): $dx_\alpha/dt = -\lambda_\alpha (\partial \mathcal{H} / \partial \xi_\alpha) + \eta_\alpha$, with $\langle \eta \rangle = 0$ and $\langle \eta_\alpha(t) \eta_\beta(t') \rangle = 2\lambda_\alpha T_\alpha \delta_{\alpha\beta} \delta(t - t')$. Note that, in the absence of coupling ($k_\times = 0$) or $T_1 - T_2 = 0$, this system will settle into thermal equilibrium. Otherwise, it is precisely an LGM with (6) in (5). The 2×2 matrices \mathbb{A} and \mathbb{D} are readily identified, while \mathbb{C}_0 and \mathbb{A} can be easily computed. In the latter, there is a single independent component, say, Λ_{12}^* , which is proportional to $k_\times (T_1 - T_2)$. Thus, we see that $L_{12}^* = \Lambda_{12}^*$ is non-zero if and only if $k_\times \neq 0$ and $T_1 \neq T_2$, and the system then settles into a NESS.

So far, our study of the PAM has led us to two point correlations at (almost) equal times, since $\mathbf{u} = d\mathbf{x}/dt = \lim_{\varepsilon \rightarrow 0} [\mathbf{x}(t + \varepsilon) - \mathbf{x}(t)] / \varepsilon$. There is a natural generalization to correlations at arbitrary times $t \neq t'$. Though rarely considered in classical mechanics of point particles, this generalization takes the form

$$\mathbf{A}(t, t') \equiv m\mathbf{r}(t) \times \mathbf{r}(t') \quad (17)$$

for any given trajectory $\mathbf{r}(t)$. Note that the magnitude $|\mathbf{A}|$ is the area of a parallelogram spanned by the two \mathbf{r} 's (related to the area in Kepler's second law). As $t' \rightarrow t_+$, \mathbf{L} is recovered: $\mathbf{L}(t) = \partial \mathbf{A}(t, t') / \partial t' |_{t'=t}$. The statistical mechanics analog of \mathbf{A} is Levy's stochastic area¹⁰. Here, we consider the two point correlation function at unequal times, $C_{\alpha\beta}(t, t')$. To be precise, the definition

$$C_{\alpha\beta}(t, t') \equiv \langle x_\alpha(t) x_\beta(t') \rangle = \int x_\alpha x'_\beta P(\mathbf{x}, t; \mathbf{x}', t') d\mathbf{x} d\mathbf{x}' \quad (18)$$

requires the *joint* probability distribution: $P(\mathbf{x}, t; \mathbf{x}', t') = P(\mathbf{x}, t) G(\mathbf{x}' - \mathbf{x}, t' - t)$. Here, $G(\xi, \tau)$ is the time dependent solution to the Fokker-Planck equation (2), subjected to the initial condition $G(\xi, 0) = \delta(\xi)$. The analog of $\mathbf{A}(t, t')$, i.e., the generalization of $L_{\alpha\beta}(t)$, is just the antisymmetric combination

$$\tilde{C}_{\alpha\beta}(t, t') \equiv C_{\alpha\beta}(t, t') - C_{\beta\alpha}(t, t') \quad (19)$$

so that $L_{\alpha\beta}(t) = d\tilde{C}_{\alpha\beta}(t, t') / dt' |_{t'=t}$. By construction, \tilde{C} is odd under $t \leftrightarrow t'$: $\tilde{C}_{\alpha\beta}(t, t') = -\tilde{C}_{\alpha\beta}(t', t)$, a property related to the violation of time reversal symmetry. Of course, in the steady state \tilde{C}^* is stationary, so it depends only on the difference $\tau = t' - t$. This formulation parallels that of relative dispersion in fluids [3, 6, 20].

Clearly, much more information about the dynamics of our system is encoded in \tilde{C}^* than in L^* and, naturally, computing these quantities theoretically is difficult in

¹⁰ Introduced in 1940 by Levy [26], this concept was subsequently developed in a series of articles. For a modern review and further generalizations, see, e.g., Ref. [12]. Most recently, it has been exploited in the context of noisy couple RC circuits [8, 9].

general. However, as the LGM is completely specified by the matrices \mathbb{A} and \mathbb{D} , it can be found analytically. In particular, the time-lagged covariance matrix \mathbb{C}_τ (i.e., $C_{\alpha\beta}^*(t, t + \tau)$) is related simply to \mathbb{C}_0 :

$$\mathbb{C}_\tau = \langle \mathbf{x}(t + \tau) \mathbf{x}^T(t) \rangle^* = e^{\mathbb{A}\tau} \mathbb{C}_0 \quad (20)$$

As expected, it is independent of t , due to time translational invariance of a steady-state. Instead of \mathbb{A} and \mathbb{D} , the LGM can be specified alternatively by these steady-state covariance matrices $\mathbb{C}_0, \mathbb{C}_\tau$. The advantage is that this representation is useful for constructing empirical models from data as discussed below. Further, note that $d\mathbb{C}_\tau/d\tau|_{\tau=0} = \mathbb{A}\mathbb{C}_0$ contains the full information contained in \mathbb{D} and \mathbb{A} . Thus, an alternative perspective of an LGM is to specify it by \mathbb{C}_0, \mathbb{D} and \mathbb{A} . The advantage of this representation is that all possible systems can be grouped into families with the same \mathbb{C}_0 and \mathbb{D} but different \mathbb{A} . Since \mathbb{D} is positive definite, one can always transform to coordinates where \mathbb{D} is diagonal[48]. In these coordinates, diffusion is solely in the radial direction in phase space. Then, phase space rotation is completely captured by \mathbb{A} . To emphasize, only one member of the family (the one with $\mathbb{A} = 0$) corresponds to a system in thermal equilibrium. All other members represent nonequilibrium systems (with the same pdf and diffusion) with non-trivial PAM. We will not pursue the study of \mathbb{C}_τ further here, but examples of its behavior can be found in other contexts [41, 28, 29].

In the final paragraphs of this section, we consider another important aspect of the PAM, namely, its full distribution. Focusing only on a NESS, we observe the system for a length of time and record a single trajectory: $\mathbf{x}_{obs}(t)$. Of course, physical data such as in climate science would be taken at typically integer multiples of some finite time step, ε . Then, instead of the continuous velocity, we would use the finite difference approximation: $d\mathbf{x}/dt \approx [\mathbf{x}(t + \varepsilon) - \mathbf{x}(t)]/\varepsilon$. From such time series, we can construct another series for

$$\mathbb{L}_{obs}(t) = \mathbf{x}_{obs}(t) \wedge \mathbf{x}_{obs}(t + \varepsilon) / \varepsilon \quad (21)$$

where we have used the wedge product to denote the antisymmetric part of the tensor product. Note that the extra term in the discrete approximation for the velocity plays no role, as $\mathbf{x}_{obs}(t) \wedge \mathbf{x}_{obs}(t) \equiv 0$. Meanwhile, under the assumption of ergodicity, various statistical quantities can be computed from a *time average* over the trajectory (denoted by an overline) instead of the ensemble averages discussed above, e.g.,

$$\overline{\mathbb{L}_{obs}} = \mathbb{L}^*$$

In addition to the average over the times series, we can construct a histogram, which approximates the full distribution of the quantity involved. For convenience, let us focus on a two-dimensional configuration space (which can be a subspace of higher dimensional space) and, dropping the subscript $_{obs}$, we will simply label the observed values as $x_1(t), x_2(t)$. Then, the PAM is characterized by a single (independent) quantity, say, the 1-2 element of \mathbb{L}_{obs} . We denote that time series by

$$\mathcal{L}(t) \equiv (x_1(t)x_2(t + \varepsilon) - x_2(t)x_1(t + \varepsilon)) / \varepsilon$$

and the associated histogram by $H(\mathcal{L})$. Note that, in general, $\mathcal{L}(t)$ will appear with both signs and the support of H is over the entire line $(-\infty, \infty)$. Normalizing H provides us with a pdf, which can be compared to the theoretical expression

$$f(\mathcal{L}) = \int \delta\left(\mathcal{L} - \frac{x_1 x'_2 - x_2 x'_1}{\varepsilon}\right) P^*(\mathbf{x}, 0; \mathbf{x}', \varepsilon) d\mathbf{x} d\mathbf{x}'$$

where $P^*(\mathbf{x}, 0; \mathbf{x}', \varepsilon) = p^*(\mathbf{x}) G(\mathbf{x}' - \mathbf{x}, \varepsilon)$ is the joint probability in NESS. In the LGM, G is also Gaussian, like p^* , so that the Fourier transform of f

$$\hat{f}(\phi) = \int f(\mathcal{L}) e^{-i\mathcal{L}\phi} d\mathcal{L} \quad (22)$$

involves only Gaussian integrals and can be computed exactly. The technical details are quite involved and will be deferred to another publication. Let us summarize the main results here.

- $1/\hat{f}(\phi)$ is the square root of the determinant of the matrix appearing in the Gaussian.
- As the matrix is 4×4 , $1/\hat{f}^2$ is a quartic polynomial in ϕ .
- The singularities of $\hat{f}(\phi)$ are branch points. Located at the roots of the quartic, they lie on both sides of the real axis, with those nearest to the real axis controlling the large \mathcal{L} asymptotic (exponential) decay of $f(\mathcal{L})$.
- The parameters of the quartic come from the defining matrices of the LGM: either the pair (\mathbb{A}, \mathbb{D}) , or the set $(\mathbb{C}_0, \mathbb{D}, \mathbb{A})$. For our $N = 2$ case, it is more convenient to use the latter set, as the first two matrices are real symmetric. Meanwhile, let us write \mathbb{A} as $\begin{pmatrix} 0 & \ell \\ -\ell & 0 \end{pmatrix}$ so that *all* the DB violating aspects of this LGM are contained in a single parameter: ℓ .
- As expected, in the $\ell = 0$ case, $1/\hat{f}^2$ is quadratic in ϕ^2 . Thus, the distribution f is symmetric in \mathcal{L} and leads to $\langle \mathcal{L} \rangle \equiv 0$.
- \hat{f} is of the form $1 - i\phi\ell + O(\phi^2)$ and so, $\langle \mathcal{L} \rangle = id\hat{f}/d\phi|_{\phi=0}$ is just ℓ , confirming Eqn. (16).
- Note that f is not δ distributed and has a finite variance. For that, we need $\langle \mathcal{L}^2 \rangle = -d^2\hat{f}/d\phi^2|_{\phi=0}$. In particular, even with $\ell = 0$, this variance, which we denote by σ_0^2 , is non-trivial. The physics is clear: A trajectory for a system in an equilibrium steady state is just as likely to rotate one way as the other, respecting time reversal symmetry. The typical values of these rotations are $O(\sigma_0)$, associated with both the damping and the noise (\mathbb{A} and \mathbb{D}).
- For $\ell \neq 0$, we find $\langle \mathcal{L}^2 \rangle = \sigma_0^2 + 2\ell^2$. As a result, we arrive at a simple expression

$$\sigma_\ell^2 \equiv \langle \mathcal{L}^2 \rangle - \langle \mathcal{L} \rangle^2 = \sigma_0^2 + \ell^2$$

for the variance of the distribution f for systems in NESS. This leads to an important ratio

$$\frac{\langle \mathcal{L} \rangle}{\sigma_\ell} = \sqrt{\frac{\ell^2}{\sigma_0^2 + \ell^2}} \quad (23)$$

which implies the following caution. If a trajectory with finite time steps is used to find averages and standard deviations of probability angular momenta, and if a NESS system is well described by an LGM, then $\langle \mathcal{L} \rangle$ can never exceed σ_ℓ . Thus, we must examine the statistics of the full pdf in order to come to a meaningful conclusion on whether a nonzero time average \mathcal{L} is significant or not. In contrast, stochastic processes with prominent rotational aspects (e.g., noisy limit cycles) are not subjected to the limitations shown here [54]. A comprehensive discussion is beyond the scope of this work and will be presented elsewhere.

Within the context of the LGM, we presented a complete analytic description of various aspects of the PAM. For systems that display prominent rotations, there is little need to identify them as NESS. But, there are many cases where the trajectories in configuration space display subtle rotations, hidden behind a substantial amount of noise. For data from computer simulations or observations, we are necessarily restricted to a finite times series of discrete points. From such a series, we may construct $\mathbb{L}_{obs}(t)$ according to (21) and compute the time average $\overline{\mathbb{L}_{obs}}$. A non-zero value is clear signal of time reversal violation, so that the system cannot be regarded as “in thermal equilibrium.” We should be cautious, however, since the variance of $\mathbb{L}_{obs}(t)$ is typically non-trivial, even in equilibrium. For an LGM with just two variables, we are able to find exact analytic results, so that these remarks rest on a sound quantitative foundation. In particular, we computed the full distribution of \mathcal{L} (the single independent quantity associated with \mathbb{L}) and showed that it is non-vanishing on both sides of $\mathcal{L} = 0$, while a non-zero average depends on the subtle asymmetry of this distribution. Given a times series, $\mathbf{x}_{obs}(t)$, a histogram can be compiled for the associated $\mathcal{L}(t)$, and we can compare that to the theoretical distribution. Such a comparison provides a further criterion, beyond fitting a Gaussian to the histogram of \mathbf{x}_{obs} , for whether the data can be adequately described by an LGM.

Finally, we address a natural question, namely, how do we interpret the sign of the PAM. Our proposal is that, especially in cases of “subtle displays” of this rotation, one of the variables is the “driver” with the other being the “follower,” much like the increase of prey populations “drives” the increase in the numbers of predators. In physical systems, the sign of \mathcal{L} may point us to more tractable underlying causes of this “driver-follower” behavior, a key characteristic of NESS. In the paleoclimate literature, a similar interpretation is used with “phase wheels” used to indicate the ordering of phenomena and likely causality under cyclical forcing, [14], here the cycles result from the PAM. In the next section, we apply these finding to two prominent examples of natural variability in the climate system. Of course, like most physical systems, the dynamics here is far from being linear and the stochastics are more complex than additive Gaussian noise. Nonetheless, for reasons that are not fully understood, it is often the case that some aspects of climate oscillations about a steady-state are skillfully modeled by LGMs.

4 Example Climate Oscillations: ENSO and MJO

Climate oscillations are preferred spatio-temporal patterns of natural variability of the climate system. These climate oscillations represent some of the most significant

variability about the time-average climate state and have important human impacts. Each oscillation has a typical fairly narrow range of timescales and has a large projection onto different, relatively small subspaces of the massively high-dimensional phase space of the entire climate system. Climate oscillations are quantified with climate indices: functions of subsets of climate variables, filtered to the specific spatio-temporal scales of the pattern, and empirically developed to capture the dominant features of a climate oscillation with one, two, or a few scalar quantities. Generally, a variety of sets of climate indices can be used to quantify a climate oscillation with mostly consistent results [51]. Oscillations often have many coevolving indices, each of which highlights a different aspect of the complex pattern.

The El-Niño Southern Oscillation (ENSO) has its largest projection in the tropical Pacific region, dominating ocean temperatures, the location of atmospheric convection and precipitation, and the atmospheric Walker circulation. Individual ENSO events (as defined by indices retaining a value beyond a critical threshold) persist for roughly 9 months and the time between repeated events is on the order of 2 to 7 years. We adopt the common description of ENSO in terms of the NINO3 index (NINO3) and the depth of the 20°C isotherm in the tropical Pacific (d20), which roughly indicates the depth of the thermocline which is the region below the surface layer where temperatures begin rapidly decreasing toward abyssal values. ENSO index data is publicly available from a number of sources. Here we use the data from the KNMI Climate Explorer [44]. The data used here are monthly averages of observations and extends from 1960 to 2016.

The Madden-Julien Oscillation (MJO) is an eastward moving pattern that has its largest projection on tropical rainfall, convection, and outgoing long-wave radiation. Its timescale is weeks to months. We will describe the MJO in terms of the so-called Original MJO Index (OMI) which is a two-dimensional index representing the principal components of the first two empirical orthogonal functions (EOFs) of filtered outgoing long-wave radiation [15]. Like ENSO data, MJO data is publicly available from a number of sources. Here we use the daily data from NOAA’s Earth System Radiation Lab 1/1/1979 - 4/26/2016.

The units of phase space for climate indices can be unintuitive. The units of probability angular momentum are the same as the units of diffusion, $\text{length}^2/\text{time}$. The two ENSO indices, however, have different units of “length”. NINO3 is a temperature anomaly and has units of °C, while d20 is a depth anomaly and has units of cm. As a result, in the NINO3-d20 phase space with monthly data the probability angular momentum has units of °C cm/month¹¹. Sometimes one uses indices that have been scaled by their standard deviation resulting in indices which are unitless (Mahalanobis distance). Then, the probability angular momentum would have units of 1/time.

Climate oscillations can be modeled by LGMs of the form (5 with 6) through a process called linear inverse modeling. A multivariate time series is used to construct the steady-state covariance \mathbb{C}_0 and the time-lagged covariance \mathbb{C}_τ . The time lag is empirically chosen to capture the timescale of the climate oscillation of interest. Then Eq. (20) is used to compute \mathbb{A} and Eq. (12) or (9) determines \mathbb{D} . There is no guarantee

¹¹ Note that “diffusion” in this case also carries these units, as it is the noise covariance matrix.

that this procedure will result in a stable SDE, but it often works surprisingly well, e.g., [50, 17, 11, 2]. Here we use linear inverse modeling to construct two-dimensional LGMs for ENSO and the MJO from the time series of their indices.

The pdf $f(\mathcal{L}_\tau)$ can be calculated directly from the observed time series as well as theoretically from the LGMs. The pdfs are strongly asymmetric and have exponential tails. The asymmetry leads to the total probability angular momentum of the steady-state, $\langle \mathcal{L}_\tau \rangle$, being nonzero despite the most likely value (mode) being zero. The two methods of generating a pdf (measurement of $f(\mathcal{L}_\tau)$ versus LGM based on \mathbb{A}, \mathbb{D}) agree surprisingly well for the ENSO and MJO cases, despite the underlying complexity of these phenomena and their quantifying indices. The linear inverse modeling procedure produces, by construction, a LGM that has same average $\langle \mathcal{L}_\tau \rangle$ as the data. Thus, we rely on the excellent agreement between the full pdf's (Fig. 2) to conclude that measurements of $f(\mathcal{L}_\tau)$ are robust and that they can be used to verify models (both LGMs and more complex climate models) as well as for model intercomparison. The two examples shown here were minimal, involving only two degrees of freedom, but the approach and equations presented here can be applied to more detailed systems as well, keeping (23) as a guide for how many probability angular momenta (involving various pairs of axes) are reliable.

5 Summary and Outlook

In this work, we study a principal characteristic of nonequilibrium stationary states, namely, *persistent* probability current loops. Of the many observable consequences, we propose to focus on a particularly convenient quantity: the probability angular momentum (in analogy with angular momenta associated with circulating fluids). Directly related to correlation functions at unequal times, it can be used to characterize any statistical system in NESS. Here, we provide an illustration in the context of climate science, i.e., the two ‘‘oscillatory’’ phenomena, ENSO and MJO.

Exploiting the parallel with fluid dynamics, we regard the probability density $P(\mathbf{x}, t)$ (in the space of configurations of a statistical system, or phase space) as a fluid density $\rho(\mathbf{r}, t)$ (in ordinary 3-dimensional space). Similarly, we draw the parallel between current *densities* $\mathbf{K}(\mathbf{x}, t)$ and $\mathbf{J}(\mathbf{r}, t)$, as well as velocity fields $\mathbf{u} = \mathbf{K}/P$ and $\mathbf{v} = \mathbf{J}/\rho$. From these, we propose to study the *probability angular momentum density*: $\mathbb{L}(\mathbf{x}, t) \equiv \mathbf{x} \wedge \mathbf{u}P$. After our system settles into a stationary state, described by distribution $P^*(\mathbf{x})$, there will be non-trivial steady \mathbf{K}^* , provided it is a NESS. These current loops lead naturally to the concept of the (total) probability angular momentum: $\mathbb{L}^* = \int d\mathbf{x} \mathbb{L}^*(\mathbf{x}) = \langle \mathbf{x} \wedge \mathbf{u} \rangle^*$. To emphasize, this quantity must vanish for a system in thermal equilibrium, in which $\mathbf{K}^* \equiv 0$ and thus, serves as a quantitative measure for the nonequilibrium characteristics of a stationary system. Since \mathbf{u} is associated with \mathbf{x} at an infinitesimally later time, \mathbb{L}^* can be generalized to two point correlation functions at arbitrary unequal times $\langle \mathbf{x}(t) \wedge \mathbf{x}(t + \tau) \rangle^* / \tau$. Noting that \mathbb{L}^* and \mathbb{D} share the same units, we find an intimate relationship between angular momentum and diffusion, the deeper significance of which is yet to be explored. In general, we argue that the probability angular momentum plays a central role in any study of the nonequilibrium fluctuations in a NESS.

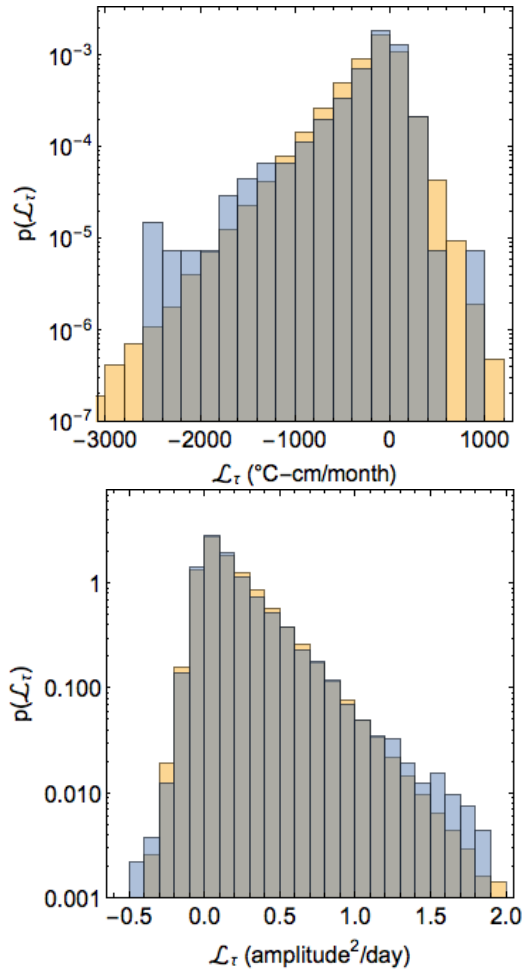


Fig. 2 Pdfs of the finite-time probability angular momentum for observations and a linear Gaussian model to the observations. a) ENSO using monthly data, and b) MJO using daily data. Gray indicates regions where the two pdfs overlap, yellow indicates regions where the model pdf is larger than the observation pdf, and blue indicates regions where the observation pdf is larger than the model pdf.

If a system evolving in NESS is observed, a single trajectory would be recorded at typically integer multiples of some time step ε : $\mathbf{x}_{obs}(n\varepsilon)$. From this, we can construct a time series for $\mathbb{L}_{obs}(n\varepsilon) = \mathbf{x}(n\varepsilon) \wedge \mathbf{x}(n\varepsilon + \varepsilon)/\varepsilon$. Invoking ergodicity, we expect the time average of \mathbb{L}_{obs} to be \mathbb{L}^* . Further, we can study its variations and compile a histogram and study the full distribution of \mathbb{L}_{obs} . In general, the variance is non-trivial, even if the time average vanishes. As a result, some care is needed when analyzing model simulations or physical data. Finally, we provide an analytically tractable case for exploring these ideas, namely, the linear Gaussian model. In particular, explicit results are found for the general case of an LGM with just two variables (in which \mathbb{L} is specified by a single quantity \mathcal{L}).

To illustrate how this approach may be applied to climate studies, we consider certain aspects of the ENSO and MJO phenomena. Using just two indices from the physical data in each case, we show that the histograms of \mathcal{L} are quite asymmetric, and so, there is no doubt that the averages $\langle \mathcal{L} \rangle$ are non-trivial. Further, we construct LGMs using the time lagged covariance from data, and found that its predictions are in excellent agreement with the histograms. Such analysis gives us confidence that the LGMs capture the essentials of this aspect of the NESS. In the climate context, it is important to note that the predictions and projections for future climate states rely on the persistent probability currents (Fig. 1). When LGMs are used for forecasting [2], it is critical to match the probability currents as observed which are, as emphasized here, a source of long timescale predictability. Our conclusion is that probability angular momenta provide a valuable and novel route to study the time reversal violating aspects of not only our climate system but also *all* systems driven far from equilibrium in general.

Naturally, many new questions arise. One line of questions follows the applications to the climate system. Needless to say, there are many, many more climate phenomena to which we can apply this type of analysis in addition to the illustrations here – the North Atlantic Oscillation, Southern and Northern Annular Modes, variability of the Oceanic Meridional Overturning Circulation, etc. The accuracy of models of these phenomena can usefully be constrained by evaluating the probability angular momenta of the model versus those of observations. The other line of pursuit is in the realm of theory. Many issues related to the PAM within the context of the LGM remain to be explored further. A prime example is the behavior of the time-lagged correlation $\langle \mathbf{x}(0) \wedge \mathbf{x}(\tau) \rangle^*$ as a function of τ , as well as their associated distributions. General considerations and a few preliminary investigations for specific systems [41, 28] show that it rises to a maximum before decaying exponentially. What is the physics behind this peak? A comprehensive study, valid for all LGMs in arbitrary dimensional phase space, would be valuable. What are the key characteristics of the LGM that can lead us to predict which are the “driving variables” and which are the “followers?” Beyond these questions associated with the probability currents, loops, angular momenta, and rotations in phase space, we should consider their implications in a wider context. Can such considerations lead us to a quantity, or quantities, beyond the twin pillars of equilibrium statistical mechanics: energy and entropy? From these steps, we may find hints towards formulating a framework for preferred fluctuations in non-equilibrium statistical mechanics and the climate system.

Acknowledgements This work was supported in part by NSF INSPIRE Award #1245944 and NSF DMR #1507371. Work of JBW was partially carried out during a stay at the Institute for Marine and Atmospheric research Utrecht (Utrecht University, NL) which was supported by the Netherlands Centre for Earth System Science. Three of us (JBW, BF-K, RKPZ) are grateful for the hospitality of the MPIPKS, where some of this work is carried out during a workshop *Climate Fluctuations and Non-Equilibrium Statistical Mechanics: an Interdisciplinary Dialogue* in the summer of 2017. We would like to thank Kevin Bassler, Ronald Dickman and Beate Schmittmann for helpful discussions.

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