Fall 2019 GEOL2300–Homework 1

1 Continuous, Piecewise Continuous, and Taylor Series

Consider the following functions (or technically, distributions), the first similar to the Heaviside function, the second the absolute value, and the third a parabola:

$$H(x) = \begin{cases} 0, & x < 0 \\ \text{undefined}, & x = 0, , \\ 1, & x > 0. \end{cases} \tag{1}$$

$$A(x) = |x|. \tag{2}$$

$$S(x) = x^2. \tag{3}$$

Recall the expression for the Taylor series located at point a:

$$f(x;a) = \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!} f^{(n)}(a). \tag{4}$$

where $f^{(n)}(a)$ implies the nth derivative of the function f evaluated at the point a.

1.1 Getting

Write out the Taylor series to order $n = 3$ of each function centered at ϵ, a small number just above 0, and at $-\epsilon$, a small number just below 0.

- a+) $H(x;\epsilon) \approx$
- a-) $H(x;-\epsilon) \approx$
- b+) $A(x;\epsilon) \approx$
- b-) $A(x;-\epsilon) \approx$
- c+) $S(x;\epsilon) \approx$
- c-) $S(x;-\epsilon) \approx$

1.2 To the Point

Evaluate the Taylor series from the preceding question at $x = 0$.

- a+) $H(0;\epsilon) \approx$

- a-) $H(0; -\epsilon) \approx$
- b+) $A(0; \epsilon) \approx$
- b-) $A(0; -\epsilon) \approx$
- c+) $S(0; \epsilon) \approx$
- c-) $S(0; -\epsilon) \approx$

1.3 Question

How does this inform your understanding of *continuous*?

Are any of the results above importantly different if you mulitiply the functions by a constant? If you add two of them together?

2 Vector Basis

Consider the vectors $\mathbf{v} = 1\mathbf{i} + 1\mathbf{j} + 1\mathbf{k}$ and $-\mathbf{v} = -1\mathbf{i} + -1\mathbf{j} + -1\mathbf{k}$, and $\mathbf{u} = 1\mathbf{i} + 1\mathbf{j}$.

2.1 Construction

What does it mean that $\mathbf{i}, \mathbf{j}, \mathbf{k}$ form a basis for $\mathbf{v}, -v, \mathbf{u}$?

2.2 Destruction

Write an equation that constructs $\mathbf{k}$ from $\mathbf{v}, -v, \mathbf{u}$

2.3 Distance

Which is longer: $\mathbf{v}$, $-v$, or $\mathbf{u}$? Which is closer to $\mathbf{v}$, $-v$, or $\mathbf{u}$?

2.4 Choose Another!

Can you find a basis that spans $\mathbf{v}, -v$, and $\mathbf{u}$ but includes only 2 vectors?

3 Functions of Functions

3.1 All to All

Provide an example of a functions that maps the whole of the real numbers to all of the real numbers and it's inverse that maps from all real numbers to all real numbers.

3.2 All to Some

Provide an example of a function that maps the whole of the real numbers to only the positive real numbers.

3.3 Combinations

If you combine the functions above, e.g., $f(g(x))$, that the answer is always a function? How do the range (output of the function) and the domain (input of the function) respond to these combinations?

3.4 Vectors?

In the previous problem, we used a 3-vector basis and a 2-vector basis to express $\mathbf{v}, -v$, and $\mathbf{u}$. What is the span (i.e., the range of all linear combinations) of the 3-vector basis? What is the span of the 2-vector basis?