Fall 2019, GEOL2300- Homework 2

1. Trigonometric and Fourier Series

1.1 The Fourier filter

The Fourier Series development of a function *f(x)* over the interval [0,a] is given by

$$f(x) = \frac{1}{\sqrt{a}} \sum_{k=-\infty}^{\infty} \widehat{f_k} e^{i2\pi kx/a}$$

- Show that the Fourier basis functions $e^{i2\pi kx/a}$ form an orthogonal set of functions where the inner product is defined as
  $$\langle f, g \rangle = \int_0^a f(x) g^*(x) dx,$$
  and g*(x) is the complex conjugate of g(x).
- Using orthogonality demonstrate that
  $$\widehat{f_k} = \frac{1}{\sqrt{a}} \int_0^a f(x)\, e^{-i2\pi kx/a} dx.$$
- Use this formula to decompose the complex exponential into a sine and cosine (Euler-Moivre formula).
- Discuss the form of the Fourier series expansion of f(x)=x over the interval [-a,a] and the first-order Taylor series expansion of sin(x), sin(2x), cos(x) and cos(2x).

2.1 Differential Equations and Fourier series

We will consider the Poisson equation in 2-D (x,y), we have a scalar field E(x,y) that we want to solve for, and another one ρ(x,y) that we know the definition of

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} = \rho(x, y)$$

- Is this PDE linear? What order is it? Is it homogenous?
- For 2-D problems the Fourier series of a function of 2 variables f(x,y) defined over [0,a]x[0,b] is
  $$f(x, y) = \frac{1}{\sqrt{ab}} \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \widehat{f_{k,l}} e^{i2\pi kx/a} e^{i2\pi lx/b}$$
  Using Fourier Series, show that Poisson equation reduces to a simple set of algebraic equations for the Fourier coefficients of f(x,y) and ρ(x,y).

3.1 The second law of thermodynamics and the condition for positive diffusion coefficients

We have the 1-D diffusion equation defined by

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}$$

Use a Fourier series decomposition of the function u(x,t) such that

$$u(x, t) = \sum_{k=-\infty}^{\infty} A_k(t) e^{ikx}$$

- Can you show that the second law of thermodynamics requires D>0?