Fall 2019, GEOL2300- Homework 3

1. **Solution to ODEs**

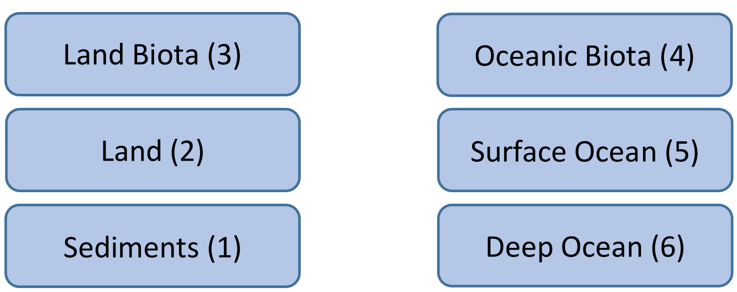
1.1 The phosphate cycle

Consider the box model of the phosphate cycle. The masses of the reservoirs are listed below

1. Sediments= 4×10^9
2. Land= 2×10^5
3. Terrestrial biota= 3000
4. Oceanic biota= 138
5. Surface ocean= 2710
6. Deep ocean= 8.71×10^4

Regarding the given fluxes below, the system is at its **steady state**, i.e. at this point, masses of the reservoirs will stay constant and won't change with time. (Fluxes have units of mass/year)

$$F_{1-2} = 20 \qquad F_{3-2} = 63.5 \qquad F_{5-6} = 17.7$$
$$F_{2-1} = 18.3 \qquad F_{4-5} = 558 \qquad F_{6-1} = 1.7$$
$$F_{2-3} = 63.5 \qquad F_{4-6} = 42 \qquad F_{6-5} = 58$$
$$F_{2-5} = 1.7 \qquad F_{5-4} = 1040$$



- Use a linear approximation for the fluxes such that $F_{A\text{-}B} = k_{A\text{-}B} A(t)$, where *k*'s are rate constants (1/time) and A(t) is the mass of phosphate in the reservoir *A* at time *t*. Write the evolution of the mass of each reservoir over time as a set of linear ODEs, where $\frac{d\boldsymbol{M}}{dt} = \mathrm{K}\boldsymbol{M}$, where **M** is an array containing the mass of each reservoir at time t (dimensions 6x1) and K is a 6x6 matrix. Write down the matrix K in terms of rate constants k's.
- Decompose **M** into a basis generated from the eigenvector $\mathbf{v}_j$ of K and plug back into the system of differential equations to get an analytical solution for this box model.
- Compare your analytical solution (plot it against …) to the numerical solution from the code that we sent you. In that code, the Land reservoir is initially perturbed to 110% of its steady state mass.Try different initial perturbations (magnitude and reservoir) to check your analytical solution.
- What do the eigenvalues of K represent? Discuss how you interpret them.

2.1 Forced harmonic oscillator

We will consider a damped harmonic oscillator with forcing

$$m\ddot{x} + \mu x + kx = F\cos wt,$$

where *m* is the mass of the oscillator, μ a damping coefficient ($\mu \geq 0$) and k the spring constant.

- What order is this ODE? Is it homogenous?
- Find the analytical solution to the non-damped case (μ=0) for the case $\frac{k}{m} \neq w^2$.
- How many characteristic timescales do you have in this non-damped scenario? What are they?
- Consider now the damped case (μ>0). Can you show that the generic function $x(t) = A\cos w_0 t + B\sin w_0 t$, with $w_0 = \sqrt{\frac{k}{m}}$, is no longer solution to the homogeneous damped oscillator equation.
- A choice for the homogenous solution to the damped oscillator equation is $x(t) = Ae^{-Bt}\cos w_0 t$. Explain the logic behind the exponential prefactor, why does it make sense? This homogenous solution introduces a new timescale in the solution 1/B, you can either derive mathematically the expression for B or BETTER use logic and your physics intuition to get it. How many characteristic timescales do we have in the problem
- **Bonus**: Show that $x(t) = G\cos wt + H\sin wt$ is a particular solution to the forced damped oscillator, what expression do you find for *G* and *H*? How does the damping term affect resonance?