Fall 2019, GEOL2300- Homework 4

1. **Tensors**

1.1 Tensor manipulations

Use Einstein notation convention and derive the following identities using the Levi-Civitta tensor

$$\varepsilon_{ijk} = \begin{cases} 0\ if\ repeated\ indices \\ +1\ if\ i,j,k\ is\ an\ even\ permutation\ of\ 1,2,3. \\ -1\ if\ it\ is\ an\ odd\ permutation\ of\ 1,2,3 \end{cases}$$

- $\nabla \cdot (\nabla \times \boldsymbol{v}) = \varepsilon_{ijk} \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^i}} \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^j}} v_k = 0$
- $\nabla \times \nabla f = 0$

2.1 Covariant description of the velocity vector in spherical coordinates

Let's define the velocity vector in Cartesian coordinates **x**=(x,y,z) as $\boldsymbol{v} = (\dot{x}, \dot{y}, \dot{z})$, then consider a spherical coordinate system **x**'=(r,θ,ϕ)

- Write down the functional forms of the mapping of (x,y,z) to (r,θ,ϕ)
- Derive the covariant version of the velocity vector in spherical coordinates using that $v'_j = \frac{\partial x^i}{\partial x'^j} v_i$, where summation is implied and primes refer to the spherical coordinate system.
- In terms of units, how does the covariant form of the velocity vector in spherical coordinates looks like? Any surprises?
- Note: the pseudo-vector for the velocity in spherical coordinates $\boldsymbol{v}' = (\dot{r}, r\dot{\theta}, r \sin\theta\, \dot{\phi})$ does not transform like a tensor (either covariant or contravariant), it is therefore a poor description of the velocity field if one strives for physical invariance among coordinate systems !

3.1 Tensorial invariance of Darcy's law

Darcy's law is an empirical law that describe the volumetric flux of fluids through a porous medium, here considering only pressure gradient (neglecting gravity here for simplicity). It is stated in the following way $\mathbf{v} = -\mathrm{K}\, \nabla p$, where K is a second-rank tensor that is the ratio of the permeability tensor and the fluid shear viscosity (scalar here).

Note: If **x** and **x**' are two coordinate systems and **v**, **v**' the respective volumetric flux in these systems, show using the rule above (2.1) for vectors, that mixed second-rank tensors transform like

$\mathrm{K}'^{m}_{n} = \frac{\partial x^i}{\partial x'^n} \frac{\partial x'^m}{\partial x^j} \mathrm{K}^j_i$ and finally the general rule between two coordinate systems that

$$\frac{\partial x^i}{\partial x'^m} \frac{\partial x'^m}{\partial x^j} = \delta^i_j = \frac{\partial x^m}{\partial x'^j} \frac{\partial x'^i}{\partial x^m}$$

- Show that Darcy's law is a proper tensorial law (invariant with change in coordinate system, i.e. $\mathbf{v}' = -\mathrm{K}'\, \nabla' p$