Part 3: Derivatives & Integrals are linear operators

Let $f : \mathbb{R}^n \longmapsto \mathbb{R}^n$ & $\vec{x}, \vec{y} \in \mathbb{R}^n$

f is a linear operator if

(i) $f(\lambda \vec{x}) = \lambda f(\vec{x})$, $\lambda \in \mathbb{R}$

(ii) $f(\vec{x} + \vec{y}) = f(\vec{x}) + f(\vec{y})$

Take a rotation around $\hat{e}_3$ in 3-D (angle θ) clockwise

$$R_3 = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

take $\vec{x} = \hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$; $\vec{y} = \hat{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$R_3 \vec{x} = \begin{pmatrix} \cos\theta \\ -\sin\theta \\ 0 \end{pmatrix}$; $R_3 \vec{y} = \begin{pmatrix} \sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$

$$R_3(\vec{x} + \vec{y}) = R_3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta + \sin\theta \\ -\sin\theta + \cos\theta \\ 0 \end{pmatrix} = R_3\vec{x} + R_3\vec{y}$$

$$R_3(\lambda \vec{x}) = R_3 \begin{pmatrix} \lambda \\ 0 \\ 0 \end{pmatrix} = \lambda R_3 \vec{x}$$

$\hookrightarrow$ R_3 is a linear operator.

Derivatives

Assume $f(x) = \lambda g(x)$ where f, g are differentiable & $\lambda \in \mathbb{R}$ is a constant

$$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$

$$= \lim_{h \to 0} \left[\frac{\lambda g(x+h) - \lambda g(x)}{h} \right] = \lambda \lim_{h \to 0} \left[\frac{g(x+h) - g(x)}{h} \right]$$

$$= \lambda g'(x)$$

Also, if $f(x) = g(x) + k(x)$ $\quad g, k$ differentiable

$$f'(x) = \lim_{h \to 0} \left\{ \frac{g(x+h) + k(x+h) - g(x) - k(x)}{h} \right\}$$

$$= \lim_{h \to 0} \left[\frac{g(x+h) - g(x)}{h} \right] + \lim_{h \to 0} \left[\frac{k(x+h) - k(x)}{h} \right]$$

$$= g'(x) + k'(x)$$

$\Rightarrow$ linear operator.

What does it mean?

Note 1:

let $\mathcal{L} u = 0$ a differential equation with $\mathcal{L}$ a linear differential operator applied to $u(x,t)$

ex, diffusion

$$\mathcal{L} \equiv \frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2}$$

Then rescaling the function $u(x,t)$ by a constant

$$v(x,t) = \lambda\, u(x,t)$$ leads to

$$\mathcal{L}v = \lambda \underbrace{\mathcal{L}u}_{=0} = 0$$ so the same differential equation applies as well.

Note 2:

Let $\mathcal{L}v = 0$ a differential equation (linear) (with $\begin{matrix}BC\text{'s}\\IC\text{'s}\end{matrix}$ defined!)

If v_1 is a function that is a solution to $\mathcal{L}v_1 = 0$ & v_2 is a different function that is also solution to $\mathcal{L}v_2 = 0$

Then $$\mathcal{L}(\underbrace{v_1 + v_2}_{v_3}) = \mathcal{L}(v_1) + \mathcal{L}v_2 = 0$$

v_3 is also a solution, & so is λv_3

=> can superpose solutions ... we have seen that

Remember $$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$$ $u(0,t) = u(1,t) = 0$
$u(x,0) = f(x)$

each $u_k = A_k \exp(-k^2 D t) \sin(\pi k x)$

is a solution $\frac{\partial u_k}{\partial t} = D \frac{\partial^2 u_k}{\partial x^2}$

Note 3: Even if derivatives are linear operators, it doesn't prevent a differential equation from being non-linear, e.g. Navier-Stokes, Burgers eq.

→ ex. $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0$

is not a linear PDE

$\left(\frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial u^2}{\partial x} = 0 \right)$ u^2 dependence of 2nd term.

if $\mathcal{L} u = \frac{\partial}{\partial t} + u \frac{\partial u}{\partial x} = 0$, then $\mathcal{L}$ is not a linear operator &

$$\mathcal{L}(u_1 + u_2) \neq \mathcal{L}(u_1) + \mathcal{L}(u_2)$$

Try it out

<u>Integrals:</u>

Let $f(x) = \lambda g(x)$

$$\int f(x)\,dx = \int \lambda g(x)\,dx = \lambda \int g(x)\,dx$$

and if $f(x) = g(x) + h(x)$

$$\int f(x)\,dx = \int [g(x) + h(x)]\,dx = \int g(x)\,dx + \int h(x)\,dx$$

$\Rightarrow$ linear operator as well.

Is it surprising? Not really, think about Fundamental theorem of calculus, if derivative is linear ....

Let's consider a system of ordinary differential (& linear) equations to highlight another approach to solve differential equations.

$$\frac{dM_1}{dt} = f_1(t, M_1, \ldots, M_N)$$

$$\frac{dM_2}{dt} = f_2(t, M_1, \ldots, M_N)$$

$$\frac{dM_3}{dt} = f_3(t, M_1, \ldots, M_N)$$

$$\vdots$$

$$\frac{dM_N}{dt} = f_N(t, M_1, \ldots, M_N)$$

Assuming that these equations track the mass of an element / chemical species in a "reservoir" over time, the

$f_1, \dots, f_N$ are balances between fluxes coming in & going out of the reservoir

e.g. f_1 = Fluxes to reservoir 1 - Fluxes out of reservoir 1.

Assuming fluxes can be linearized so that

$$F_{A \to B} = k_{A,B} \, A(t)$$

flux from reservoir $A \to B$

$k_{A,B}$ is a rate.

$$\frac{dM_1}{dt} = -\sum_{j \neq 1} k_{1,j} M_1 + \sum_{j \neq 1} k_{j,1} M_j$$

$$\frac{dM_2}{dt} = -\sum_{j \neq 2} k_{2,j} M_2 + \sum_{j \neq 2} k_{j,2} M_j$$

$$\vdots$$

N equations

N unknowns (M_j)

$$\vec{M} = \begin{pmatrix} M_1(t) \\ \vdots \\ M_N(t) \end{pmatrix}$$

System of equations can be recast as:

$$\frac{d\vec{M}}{dt} = \mathbb{K}\vec{M}$$

where $\mathbb{K}$ is a $N \times N$ matrix of the form

$$\mathbb{K} = \begin{pmatrix} -\sum_{j \neq 1} k_{1,j} & k_{2,1} & k_{3,1} & \cdots & k_{N,1} \\ k_{1,2} & -\sum_{j \neq 2} k_{2,j} & & & k_{N,2} \\ \vdots & & \ddots & & \vdots \\ k_{1,N} & k_{2,N} & \cdots & & -\sum_{j \neq N} k_{N,j} \end{pmatrix}$$

We can decompose $\vec{M}$ (vector in our N-space) into a basis $\vec{V}_1, \vec{V}_2, \ldots \vec{V}_N$

so that $\vec{M} = \sum_{j=1}^{N} a_j(t) \vec{V}_j$

Any suitable set of $\vec{V}_j$'s would work (if they constitute a basis), what if we consider eigenvectors of M?

$$M \vec{V}_j = \lambda_j \vec{V}_j$$

If the operator M is real-valued (it has to!) and a square matrix & bijective (can be inverted) then

$$\sum_j a_j \vec{V}_j = 0 \Rightarrow a_j = 0 \ \forall j$$

linear independence of $\vec{V}_j$

& $\vec{V}_j$'s span the whole space (so good vectorial basis)

Back to $\frac{d\vec{M}}{dt} = \mathbb{K}\vec{M}$

$$\vec{M} = \sum_j a_j(t)\vec{v}_j \qquad (\vec{v}_j \neq \text{depend on time})$$

$$\frac{d\vec{M}}{dt} = \sum_j \frac{da_j}{dt}\vec{v}_j \quad ; \quad \mathbb{K}\vec{M} = \sum_j a_j(t)\,\mathbb{K}\vec{v}_j$$

$$= \sum_j \lambda_j a_j(t)\vec{v}_j$$

$\Rightarrow$ ODE's:

$$\sum_j \left[\frac{da_j}{dt} - \lambda_j a_j\right]\vec{v}_j = 0 \quad , \text{ linear independence of } \vec{v}_j$$

$\Downarrow$

$$\frac{da_j}{dt} = \lambda_j a_j \, .$$

$$a_j(t) = A_j \exp(\lambda_j t)$$

$$\vec{M}(t) = \sum_j A_j \exp(\lambda_j t)\,\vec{v}_j$$

, λ_j, $\vec{v}_j$ are eigenvalues & eigenvectors of $\mathbb{K}$

A_j's are determined from initial conditions

$$\vec{M}(0) = \sum_j A_j \vec{v}_j$$

Show phosphorus cycle ...