Tensors

Part 1 : Why tensors:

(a) In physics, we strive to write laws in a so-called invariant format; ie. physics doesn't depend on the choice of coordinate system.

e.g. Newton's 2^{nd} law

$$m \frac{d^2\vec{x}}{dt} = \sum \vec{F}$$

should be valid in any reference frame, even if $\vec{x}$ & $\vec{F}$ will change from a coordinate system to another.

$\Rightarrow$ we need a way to express physical laws independently of the coordinate system.

(b) tools of vector calculus are incapable of that, see the velocity vector

$\vec{V}$, in a cartesian coordinate system

$$\vec{V} = \underbrace{(\dot{x}, \dot{y}, \dot{z})}_{} = \dot{x}\,\hat{e}_x + \dot{y}\,\hat{e}_y + \dot{z}\,\hat{e}_z$$

coordinates all have the same units (length/time) & unit vectors $\hat{e}_k$ are unitless.

Change to spherical coordinates $\vec{x} = (x, y, z) \rightarrow (r, \theta, \varphi)$

$$\vec{V} \overset{?}{=} (\dot{r}, \dot{\theta}, \dot{\varphi}) \quad (*)$$

$\Rightarrow$ do not share same units

So maybe a better choice in that case is

$$\vec{V} = (\dot{r}, r\dot{\theta}, r\sin\theta\,\dot{\varphi}) \quad (**)$$

all coordinates have again units of velocity

Which one is better / correct?

A tensor can be covariant, contravariant or expressed in physical form, we will see that in more details
← not really a tensor!

Note: (*) is contravariant
(**) is in physical form (neither covariant nor contravariant)

Some arrays are not even vectors

e.g (ρ, p, T) (density, pressure, temperature)

is not a vector, just an array!
⇒ why? Doesn't follow the rules for transformation of tensors for changes in coordinate systems.

We will explore two complementary (fundamentally identical) viewpoints in discussing tensors

(A). "Linear algebra" view point
(vectorial spaces, vector basis...)

(B). Functional view point
(metric tensor, operators)

Let's start a bit with A (Algebra viewpoint):

Let $\vec{u}$ be a vector in $\mathbb{R}^n$ (or $\mathbb{C}^n$ ...)

$$\vec{u} = \sum_{\mu=1}^{n} u^{\mu} \underbrace{\hat{e}_{\mu}}_{\text{vector basis}} = u^1 \hat{e}_1 + u^2 \hat{e}_2 + \ldots + u^n \hat{e}_n$$

not power, index

The set of u^{μ} corresponds to the coordinate of $\vec{u}$ in the basis $\underbrace{\{\hat{e}_{\mu}\}_{\mu}}$ (they are unique)

this vectorial basis will be called B (coordinate system)

Let's define another coordinate system $B' \rightarrow \{\hat{e}'_{\mu}\}_{\mu}$

Where the matrix $(n \times n)$ L describes the change of reference frame:

$$\underset{\in B'}{\hat{e}'_{\mu}} = \sum_{\gamma=1}^{n} L^{\gamma}{}_{\mu} \underset{\in B}{\hat{e}_{\gamma}} \quad , \quad L^{\gamma}{}_{\mu}$$

is a matrix
γ index for rows
μ for columns.

Now, let's define $\Lambda = L^{-1}$

so that $\Lambda L = L \Lambda = \mathbb{1}$ identity matrix $(n \times n)$

In component form

$$\sum_{\nu=1}^{n} L^{\mu}{}_{\nu} \Lambda^{\nu}{}_{\rho} = \delta_{\rho}^{\mu} = \begin{cases} = 0 & \text{if } \mu \neq \rho \\ = 1 & \text{if } \mu = \rho \end{cases}$$

δ^{μ}_{ρ} is called the Kronecker delta.

Similarly

$$\hat{e}_{\mu} = \sum_{\nu=1}^{n} \Lambda^{\nu}_{\mu} \, \hat{e}'_{\nu}$$

Back to $\vec{u}$ now

$$\vec{u} = \sum_{\mu} u^{\mu} \underbrace{\hat{e}_{\mu}}_{\in B} = \sum_{\mu} u'^{\mu} \underbrace{\hat{e}'_{\mu}}_{\in B'}$$ in the two bases.

$$= \sum_{\mu,\nu} u^{\mu} (\Lambda^{\nu}_{\mu} \hat{e}'_{\nu}) \qquad = \sum_{\mu,\nu} u'^{\mu} (L^{\nu}_{\mu} \hat{e}_{\nu})$$

Matching those equations shows us how $\vec{u}$ transforms from $B \to B'$

$$\sum_{\mu,\nu} u^{\mu} (\Lambda^{\nu}_{\mu} \hat{e}'_{\nu}) = \sum_{\mu} u'^{\mu} \hat{e}'_{\mu}$$

$$\Rightarrow \quad \boxed{u'^{\mu} = \sum_{\nu} \Lambda^{\mu}_{\nu} u^{\nu}}$$

&

$$\sum_{\mu,\nu} u'^{\mu} (L^{\nu}_{\mu} \hat{e}_{\nu}) = \sum_{\mu} u^{\mu} \hat{e}_{\mu}$$

$$\Rightarrow \quad \boxed{u^{\mu} = \sum_{\nu} L^{\mu}_{\nu} u'^{\nu}}$$

Mmmh.... interesting ...

bases transform:

$$\hat{e}'_\mu = \sum_{\nu=1}^{n} L^\nu_\mu \, \hat{e}_\nu$$

but $\vec{u}$ transforms:

$$\vec{u}'^\mu = \sum_\nu \Lambda^\mu_\nu \, u^\nu$$

In that case, we will refer to $\vec{u}$ as a contravariant vector.

NOTE: Einstein summation notation → assume summation over repeated indices (more complicated than that ... but ...)

$$\Rightarrow \text{e.g. } \hat{e}'_\mu = L^\nu_\mu \, \hat{e}_\nu$$

$$u'^\mu = \Lambda^\mu_\nu \, u^\nu$$

(ρ, p, T) is not a vector because it doesn't transform in this way upon change in coordinate system

ρ, p, T are scalar quantities ⇒ independent of reference frame.

Other point of view $\rightarrow$ functional approach

Let's define two different coordinate systems

$x_i \ , \ i = 1, \dots, n,$ & $x_i' \ , \ i = 1, \dots, n$, if we now how to transform from one to the other

$$x_i' = x_i'(x_1, \dots, x_n)$$ &

$$x_i = x_i(x_1', \dots, x_n')$$

$$\left(\text{e.g.} \quad x^{\nu} = \sum_{\mu} L^{\nu}_{\ \mu} x'^{\mu} \quad \& \quad x'^{\nu} = \sum_{\mu} \Lambda^{\nu}_{\ \mu} x^{\mu} \right)$$

we can get a sense for how any tensor transforms with that change of coordinate system using the chain rule.

Let's take the gradient of a function $f(x_1, \dots, x_n)$

How is $\dfrac{\partial f}{\partial x^{\mu}}$ (μ-component of f gradient in coordinate system x_i)

related to $\dfrac{\partial f}{\partial x'^{\mu}}$? (μ - comp. ... in x_i')

$$\frac{\partial f}{\partial x'^{\mu}} = \frac{\partial f}{\partial x^{\nu}} \underbrace{\frac{\partial x^{\nu}}{\partial x'^{\mu}}}_{\text{2 independent indices (like a matrix)}}$$

well, $\dfrac{\partial x^{\nu}}{\partial x'^{\mu}} = L^{\nu}_{\ \mu}$ in our other description

That is different from $\vec{u}$ (contravariant vector), the gradient of f transforms like a covariant tensor (here, vector).

Interesting!

What about the differential of a coordinate

$$dx^{\mu} \longrightarrow dx'^{\mu} \; ?$$

$$dx'^{\mu} = dx^{\nu} \frac{\partial x'^{\mu}}{\partial x^{\nu}} = dx^{\nu} \Lambda^{\mu}{}_{\nu}$$

Like $(\vec{u})^{\mu}$, dx^{μ} transforms in a contravariant way

so $\vec{u}$, $\vec{dx}$ are contravariant vectors

grad f (∇f) is a covariant vector

& $\vec{v} = (\dot{r}, r\dot{\theta}, r\sin\theta\,\dot{\varphi})$ is not even a tensor.

In general, $L^{\mu}{}_{\nu} = g^{\mu}{}_{\nu}$ is a metric tensor (2nd rank)

and it admits a conjugate metric tensor

$$g_{\mu}{}^{\nu} = \Lambda^{\nu}{}_{\mu}$$

so that $g_{\rho}{}^{\nu} g^{\rho}{}_{\mu} = \delta^{\nu}_{\mu}$ $\qquad (\Lambda L = \mathbb{1})$

Before digging deeper into the metric tensor, let's move back to our linear algebra perspective...

Let's define a linear form acting on a vectorial space E w such that

$$w : \vec{u} \in E \longmapsto \mathbb{R}$$, w is a linear application

The action of w on $\vec{u}$ is sometimes written as

$w(\vec{u})$ or $\langle w, u \rangle$

One example we have already seen is

$w \rightarrow$ differential coordinate

$$w_i(\vec{x}) = d_i(\vec{x}) = dx^i$$

or the inner product $\langle w, u \rangle$ (scalar product) more on that soon.

The space of all linear forms w acting on E is also a vectorial space of same dimension as E, called E^* or the dual of E

$\Downarrow$

linear forms are then vectors of E^* and can be decomposed into a basis $\{\hat{\alpha}^\mu\}_\mu$

$$\vec{w} = w_\mu \hat{\alpha}^\mu$$

with w_μ being covariant coordinates of vector $\vec{w}$

If $\{\hat{\alpha}^\mu\}$ is the dual basis of $\{\hat{e}_\mu\}_\mu$
$\in E^*$ $\in E$

then for any $\vec{u} \in E$ & $\vec{w} \in E^*$

$$\vec{w} = \sum_\mu w_\mu \hat{\alpha}^\mu \overset{\text{Einstein}}{=} w_\mu \hat{\alpha}^\mu \implies w_\mu = \langle \vec{w}, \hat{e}_\mu \rangle$$

$$\& \quad \vec{u} = \sum_\mu u^\mu \hat{e}_\mu = u^\mu \hat{e}_\mu \implies u^\mu = \langle \hat{\alpha}^\mu, \vec{u} \rangle$$

That is true because $\langle \hat{\alpha}^\mu, \hat{e}_\nu \rangle = \delta^\mu_\nu$ for dual bases!

We have seen $\{\hat{e}_\mu\}_\mu \longrightarrow \{\hat{e}'_\mu\}_\mu$ with L

$$\hat{e}'_\mu = L^\nu_\mu \hat{e}_\nu$$

, how does the basis $\hat{\alpha}^\mu$ transforms into $\hat{\alpha}'^\mu$?

If $\{\hat{\alpha}'^\mu\}_\mu$ is the dual of $\{\hat{e}'_\mu\}_\mu$

then again

$$\langle \hat{\alpha}'^\mu, \hat{e}'_\nu \rangle = \delta^\mu_\nu$$

$$= \langle \hat{\alpha}'^\mu, L^\rho_\nu \hat{e}_\rho \rangle = \langle \Pi^\mu_\gamma \hat{\alpha}^\gamma, L^\rho_\nu \hat{e}_\rho \rangle$$

$$= \Pi^\mu_\gamma L^\rho_\nu \langle \hat{\alpha}^\gamma, \hat{e}_\rho \rangle = \Pi^\mu_\gamma L^\rho_\nu \delta^\gamma_\rho = \Pi^\mu_\gamma L^\gamma_\nu = \delta^\mu_\nu$$

That proves that Π is the inverse of L

$$\Rightarrow \quad \Pi = \Lambda$$

$$\hat{\alpha}'^{\mu} = \Lambda^{\mu}{}_{\nu}\, \hat{\alpha}'^{\nu}$$

$$\hat{\alpha}^{\mu} = L^{\mu}{}_{\nu}\, \hat{\alpha}'^{\nu}$$

$\Rightarrow$ basis changes in a <u>contravariant</u> way in dual space E^*

But!

$$\vec{w} = w_\mu\, \hat{\alpha}^{\mu} = w_\mu\, L^{\mu}{}_{\nu}\, \hat{\alpha}'^{\nu} = w_\mu'\, \hat{\alpha}'^{\mu}$$

$$\Rightarrow \quad \boxed{w_\mu' = L^{\nu}{}_{\mu}\, w_\nu}$$

linear forms transform like <u>covariant</u> tensors!

$$w_\mu' = L^{\nu}{}_{\mu}\, w_\nu$$

e.g. $\quad w_\mu = \dfrac{\partial f}{\partial x^\mu}$ $\quad$ f is a scalar function of x^μ $\mu = 1, .., n$

$$\frac{\partial f}{\partial x'^{\mu}} = \underbrace{\frac{\partial x^{\nu}}{\partial x'^{\mu}}}\, \frac{\partial f}{\partial x^{\nu}}$$

$L^{\nu}{}_{\mu}$ is again the metric tensor for that coordinate change.

$\Rightarrow$ Vectors can transform either in a covariant or a contravariant way upon base (coordinates) change.
