Back to "functional form" description

Important theorem :

$$\frac{\partial x^i}{\partial \tilde{x}^k} \frac{\partial \tilde{x}^k}{\partial x^j} = \delta^i_j \qquad \text{(inverse relation)}$$

=> Simple prof by the chain rule.

Tensor contraction

let a 3rd-rank mixed tensor $T_{i\,j}^{\ \ k}$ be defined,

the expression $T_{i\,j}^{\ \ j}$ is a tensor contraction

(Einstein's notation)

$$T_{i\,j}^{\ \ j} = A_i \qquad \text{(1st rank tensor - vector)}$$

A contraction of a 2nd rank tensor is a scalar & is invariant with respect to coordinate change.

$$\tilde{T}_k^{\ k} = \frac{\partial x^i}{\partial \tilde{x}^k} \frac{\partial \tilde{x}^k}{\partial x^j} T_i^{\ j} = \delta^i_j T_i^{\ j} = \underbrace{\lambda}_{\text{scalar}}$$

The metric tensor:

In a coordinate system x^i (contravariant coordinate of position in that system), the differential displacement vector $d\vec{r}$ is

$$d\vec{r} = \left(h_{(1)}\, dx^1,\ h_{(2)}\, dx^2,\ \dots,\ h_{(n)}\, dx^n \right) = \sum_{i=1}^{n} h_{(i)}\, dx^i\, \hat{e}_{(i)}$$

$\hat{e}_{(i)}$ are physical unit vectors NOT covariant ones

			x^i
$\hat{e}_{(i)}$	in cartesian	$\rightarrow \hat{e}_x, \hat{e}_y, \hat{e}_z$	(x, y, z)
	in spherical	$\rightarrow \hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi$	(r, θ, φ)

$h_{(i)}$ are scale factors (not tensors)

the length of $d\vec{r}$ or $d\vec{r} \cdot d\vec{r}$ is

$$(dr)^2 = \sum_i \sum_j h_{(i)}\, h_{(j)}\, \hat{e}_{(i)} \cdot \hat{e}_{(j)}\, dx^i dx^j \quad \text{is a scalar}$$

we will define the coefficients

$$g_{ij} \equiv h_{(i)}\, h_{(j)}\, \hat{e}_{(i)} \cdot \hat{e}_{(j)} = g_{ji}$$

$\Rightarrow$ metric g_{ij}

If the coordinate system is orthogonal

$$g_{ij} = h_{(i)}\, h_{(j)}\, \delta_{ij}$$

Back to $(dr)^2$

With the metric we have:

$$(dr)^2 = g_{ij}\, dx^i dx^j \qquad \text{(implicit summation)}$$

Notes

=> The metric is a 2nd rank covariant tensor
(not proved here)

=> The conjugate metric g^{kl} is defined as

$$g^{kl} g_{il} = \delta_i^k \qquad \text{(inverse of metric)}$$

is a contravariant 2nd-rank tensor.

Now, remember the coordinate differential (contravariant)
dx^i, we can write its covariant form

$$dx_i = g_{ij}\, dx^j \qquad \text{using the metric}$$

Similarly (given $g^{kl} g_{il} = \delta_i^k$) we have

$$dx^i = g^{ij}\, dx_j$$

=> the metric (and its conjugate) are the "elevators" for indices

We discussed that $(x,y,z) - (r,\theta,\varphi)$ coordinates

$\vec{v} = (\dot{r}, \dot{\theta}, \dot{\varphi})$ transforms like a tensor

but not $\vec{u} = (\dot{r}, r\dot{\theta}, r\cos\theta\, \dot{\varphi})$

Let $\vec{A} = \sum_i \underbrace{A_{(i)}}_{\text{physical components (same unit as } \vec{A})} \underbrace{\hat{e}_{(i)}}_{\text{unit basis (vectors) (unitless)}}$ unique (not a tensorial representation)

Remember $d\vec{r}$, let's try something similar

$$\vec{A} = \sum_i h_{(i)} A^i \hat{e}_{(i)}$$

A^i ← contravariant component

$\hat{e}_{(i)}$ → same basis (physical)

$$\Rightarrow A(i) = h_{(i)} A^i \Rightarrow A^i = \frac{1}{h_{(i)}} A(i)$$

$h_{(i)}$ is a scaling factor

now comes the metric (well its conjugate)

$$A^i = \cancel{g_{ij}}\, g^{ij} A_j$$

$$\Rightarrow A(i) = h_{(i)} g^{ij} A_j$$

$$\Rightarrow A_j = \sum_i \frac{g_{ij}}{h_{(i)}} A_{(i)}$$

=> relationship between physical & tensor coordinates (use the metric Luke ....)

Same idea for 2nd rank tensors:

$$T_{(ij)} = h_{(i)} h_{(j)} g^{ik} g^{jl} T_{kl} \quad \& \quad T_{(ij)} = h_{(i)} h_{(j)} T^{ij}$$

Now, let's get back to scalar product between two contravariant tensors A^i B^i

$$A^i B_i = \sum_i \frac{1}{h_{(i)}} A_{(i)} \sum_j \frac{g_{ij}}{h_{(j)}} B_{(j)} = \sum_{ij} A_{(i)} B_{(j)} \frac{g_{ij}}{h_{(i)} h_{(j)}}$$

Definition of metric tensor is

$$g_{ij} = h_{(i)} h_{(j)} \hat{e}_{(i)} \cdot \hat{e}_{(j)}$$

$$\Rightarrow \frac{g_{ij}}{h_{(i)} h_{(j)}} = \hat{e}_{(i)} \cdot \hat{e}_{(j)}$$

$$\Rightarrow A^i B_i = \sum_{i} \left(A_{(i)} \hat{e}_{(i)}\right) \cdot \sum_j \left(B_{(j)} \hat{e}_{(j)}\right)$$

So that's really the scalar product we know

Also

$$g^{ij} A_i B_j = A^j B_j$$

$$\& \quad g_{ij} A^i B^j = A^i B_i$$

are the same

=> scalar product yield same value for covariant & contravariant tensers.

Same idea applies to 2nd-rank tensors.

$$S^{ij} T_{ij} = \underline{\underline{S}} : \underline{\underline{T}} \quad \text{is a scalar}$$

Now, let's derive a covariant basis vector

$\hat{e}_i$ instead of a physical one

if $\vec{r}_x$ is the displacement vector whose components are expressed in coordinate system x^i

$$\hat{e}_i \equiv \frac{d\vec{r}_x}{dx^i}$$

not a "true" index for a tensor
just indicates a vector (the ith) in a collection of vectors.

New ref frame $x^{i'}$

$$\hat{e}_i' = \frac{d\vec{r}_x'}{dx^{i'}} = \frac{\partial x^j}{\partial x^{i'}} \frac{d\vec{r}_x}{dx^j} = \frac{\partial x^j}{\partial x^{i'}} \hat{e}_j$$

$\Rightarrow$ covariant.

$$\hat{e}_i = h_{(i)} \hat{e}_{(i)} \qquad \hat{e}_{(i)} = \frac{1}{h_{(i)}} \hat{e}_i$$

$$\hat{e}^j = g^{ij} \hat{e}_i \quad \Rightarrow \quad \hat{e}_{(i)} = \frac{g_{ij}}{h_{(i)}} \hat{e}^j$$

Cool stuff here, regardless of orthogonality of $\hat{e}_{(i)}$'s

$$\hat{e}_i \cdot \hat{e}^j = \delta_i^j$$

$$\underbrace{h_{(i)}\hat{e}_{(i)}}_{\hat{e}_i} \underbrace{\sum_k h_{(k)} g^{kj} \hat{e}_{(jk)}}_{\hat{e}^j} = \sum_k g^{kj} \underbrace{h_{(i)} h_{(k)} \hat{e}_{(i)} \cdot \hat{e}_{(k)}}_{\equiv g_{ik}}$$

$$= g^{kj} g_{ik} = \delta_i^j \quad \square$$

With these covariant basis vectors

$$\vec{A} = A^i \hat{e}_i = A_j \hat{e}^j$$

Let $\vec{A} = A^i \hat{e}_i$, how do we write the tensorial derivative of $\vec{A}$?

$$\frac{\partial \vec{A}}{\partial x^j} = \frac{\partial (A^i \hat{e}_i)}{\partial x^j} = \frac{\partial A^i}{\partial x^j} \hat{e}_i + \frac{\partial \hat{e}_i}{\partial x^j} A^i$$

For shorthand, we will refer to

$$\frac{\partial \hat{e}_i}{\partial x^j} \equiv \underbrace{\Gamma_{ij}^{\ \ k}}\, \hat{e}_k$$

Component of $\frac{\partial \hat{e}_i}{\partial x^j}$ in basis elmt $\hat{e}_k$

~~$\hat{e}_i$~~

$$\frac{\partial \hat{e}_i}{\partial x^j} \cdot \hat{e}^l = \Gamma_{ij}^{\ \ k} \underbrace{\hat{e}_k \cdot \hat{e}^l}_{\delta_k^l} = \Gamma_{ij}^{\ \ l}$$

so now

$$\frac{\partial \vec{A}}{\partial x^j} = \frac{\partial (A^i \hat{e}_i)}{\partial x^j} = \frac{\partial A^i}{\partial x^j} \hat{e}_i + \underbrace{\frac{\partial \hat{e}_i}{\partial x^j}}_{\Gamma^k_{ij} \hat{e}_k} A^i$$

$$\frac{\partial \vec{A}}{\partial x^j} = \left(\frac{\partial A^i}{\partial x^j} + \Gamma^i_{jk} A^k \right) \hat{e}_i$$

$$\Rightarrow \quad \nabla_j A^i \equiv \frac{\partial A^i}{\partial x^j} + \Gamma^i_{jk} A^k$$ is the covariant derivative of a contravariant vector.

Ok, now we are equiped to deal with tensor calculus & connect it to vector calculus.

f a scalar function of coordinate system x^i

∇f is the physical gradient

How do we express a gradient in tensor notation?