Back to Christoffel symbols

$$\Gamma_{i\ j}^{\ l} = \frac{\partial \hat{e}_i}{\partial x^j} \hat{e}^l$$

$$\Gamma_{ijk} = g_{kl} \Gamma_{i\ j}^{\ l} \quad \& \quad \Gamma_{i\ j}^{\ k} = g^{kl} \Gamma_{ijl}$$

Because we have

$$g_{ij} = \hat{e}_i \hat{e}_j$$

then $$\frac{\partial g_{ij}}{\partial x^k} = \frac{\partial(\hat{e}_i \hat{e}_j)}{\partial x^k} = \frac{\partial \hat{e}_i}{\partial x^k} \hat{e}_j + \frac{\partial \hat{e}_j}{\partial x^k} \hat{e}_i$$

Remembering that $\Gamma_{ijk} = g_{kl} \frac{\partial \hat{e}_i}{\partial x^j} \hat{e}^l = \frac{\partial \hat{e}_i}{\partial x^j} \hat{e}_k$

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma_{ikj} + \Gamma_{jki}$$

& playing with indices leads similarly to:

$$\frac{\partial g_{ki}}{\partial x^j} = \Gamma_{kji} + \Gamma_{ijk}$$

also $\Gamma_{ijk} = \Gamma_{jik}$

& $$\frac{\partial g_{jk}}{\partial x^i} = \Gamma_{jik} + \Gamma_{kij}$$

$$\Rightarrow \Gamma_{ijk} = \left[\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right] / 2$$

and therefore

$$\Gamma_{ij}^{\ l} = g^{lk}\,\Gamma_{ijk} = \frac{g^{lk}}{2}\left[\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right]$$

So we have seen the covariant derivative of a contravariant vector $\vec{A} = A^i \hat{e}_i$

$$\frac{\partial \vec{A}}{\partial x^j} = \underbrace{\left(\frac{\partial A^i}{\partial x^j} + \Gamma^i_{jk} A^k\right)}_{= \nabla_j A^i}\hat{e}_i = \nabla_j A^i \hat{e}_i$$

$\Rightarrow$ this transforms as a mixte tensor.

the covariant derivative of a covariant vector:

$$\vec{A} = A_i\, \hat{e}^i$$

$$\frac{\partial A_i \hat{e}^i}{\partial x^j} = \frac{\partial A_i}{\partial x^j}\hat{e}^i + \underbrace{\frac{\partial \hat{e}^i}{\partial x^j}}_{\Gamma_{ij}^{\ k}\hat{e}_k} A_i$$

$$\Rightarrow \quad \nabla_j A_i = \frac{\partial A_i}{\partial x^j} - \Gamma_{ij}^{\ k} A_k$$

Ok, let's pause an instant & summarize

In vector calculus, one can compute the gradient of a vector and obtain "an object" of rank 2 ($\sim$ matrix)

$$\nabla \vec{A} = \mathbb{M} \qquad \text{if } \vec{A} \in \mathbb{R}^n, \text{ then } \mathbb{M} \in \mathbb{R}^n \times \mathbb{R}^n$$

In tensorial calculus, the invariance of physical laws to coordinate changes influences how one thinks about differentiation (grad, div...)

$$\text{ex.} \qquad \nabla_j A^i = \frac{\partial A^i}{\partial x^j} + \underbrace{\Gamma^i_{jk} A^k}_{\text{new!}}$$

$$\text{with} \qquad \Gamma^i_{jk} = \frac{1}{2} g^{im} \left[\frac{\partial g_{km}}{\partial x^j} + \frac{\partial g_{mj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^m} \right]$$

=> the new term $\neq 0$ only if metric is not homogeneous! (Deformed space!)

↳ fine in cartesian or std spherical coordinates.

Let's go back to our linear algebra description where we learned about dual space E^*, with $W \in E^*$

$$\vec{W} = W_\mu \hat{\alpha}^\mu \qquad \text{Coordinates } W_\mu \text{ are covariant}$$

if $\vec{X} \in E$ $\qquad W_\mu = W_\mu(\vec{X})$ is a 1-form (linear)

$\hat{\alpha}^\mu$ is a contravariant basis of E^*

There is plenty of choice for the basis $\hat{\alpha}^\mu$

$w_i(\vec{x}) = dx^i$ the differential form (coordinates)

is a good choice for a basis

if $\hat{e}_\mu$ are linearly independent $\Rightarrow$ dx^i $i=1,\dots,N$

are also linearly independent

$\hat{\alpha}^\mu \longrightarrow d\hat{x}^\mu$ will be our choice of

basis for E^*

So any vector $\vec{w} \in E^*$ can be written (unique) as

notation: $\vec{w} = f_\mu \, d\hat{x}^\mu$ with f_μ linear functions $(w_\mu \rightarrow f_\mu)$

In Cartesian coordinates a 1-form can be written

$$\vec{w} = f_x \, \hat{dx} + f_y \, \hat{dy} + f_z \, \hat{dz}$$

A staple of vectorial / tensorial calculus is Stokes theorem. If the outter derivative of $w \to dw$ & γ is the geometric border of an object φ $(\gamma \equiv \partial \varphi)$

Stokes th:
$$\int_\gamma w = \int_\varphi dw$$

For 0-form $w = f(\vec{x})$ scalar fct in E^*

Stokes theorem $\to$ $\displaystyle\int_\gamma \nabla f \cdot \vec{dl} = f(\gamma(1)) - f(\gamma(0))$

Gradient theorem.

The outter derivative dw of w is an interesting object

If w is a k-form $\left(\text{e.g } E^* = \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k \text{ times}} \right)$

ex. 2-form will be an operator with 2 indices (hydraulic permeability has 2 indices)

then dw is a $(k+1)$-form

Here, we will be interested by 2-forms and will write a 2-form

$$w_2 = w_{i,j}(\vec{x}) \underbrace{dx^i \wedge dx^j}$$

think of a 2nd rank basis vector but akin to cross-product $\to$ anti-symmetric

$$dx^i \wedge dx^j = -dx^j \wedge dx^i$$

for 3-form

$$w_3 = w_{ijk}(\vec{x})\, dx^i \wedge dx^j \wedge dx^k \quad \text{(still anti-symmetric)}$$

Ok so 1-form $w = f_x\, dx + f_y\, dy + f_z\, dz$

$$dw = df_x \wedge dx + df_y \wedge dy + df_z \wedge dz$$

with $df_x = \frac{\partial f_x}{\partial x} dx + \frac{\partial f_x}{\partial y} dy + \frac{\partial f_x}{\partial z} dz$

$$\Rightarrow dw = \left(\frac{\partial f_x}{\partial x} dx + \frac{\partial f_x}{\partial y} dy + \frac{\partial f_x}{\partial z} dz\right) \wedge dx + \left(\frac{\partial f_y}{\partial x} dx + \frac{\partial f_y}{\partial y} dy + \frac{\partial f_y}{\partial z} dz\right) \wedge dy$$
$$+ \left(\frac{\partial f_z}{\partial x} dx + \frac{\partial f_z}{\partial y} dy + \frac{\partial f_z}{\partial z} dz\right) \wedge dz$$

Regrouping & using $dx \wedge dy = -dy \wedge dx$

$$dw = \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) dx \wedge dy + \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right) dy \wedge dz$$
$$+ \left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right) dz \wedge dy$$

$$= \nabla \times \vec{f}\; d^2\vec{x}$$

Back into Stokes

$$\int_{\varphi} dw = \int_{\gamma} w$$

$$\int_{S} \nabla \times \vec{f} \cdot d^2\vec{x} = \oint_{\partial S} \vec{f} \cdot d\vec{l} \qquad \text{Green's th.}$$

if w is a 2-form

$$w = f_x \, dy \wedge dz + f_y \, dz \wedge dx + f_z \, dx \wedge dy$$

$$dw = df_x \wedge dy \wedge dz + df_y \wedge dz \wedge dx + df_z \wedge dx \wedge dy$$

with $df_x = \frac{\partial f_x}{\partial x} dx + \frac{\partial f_x}{\partial y} dy + \frac{\partial f_x}{\partial z} dz$

$$dw = \left(\frac{\partial f_x}{\partial x} dx + \frac{\partial f_x}{\partial y} dy + \frac{\partial f_x}{\partial z} dz \right) \wedge dy \wedge dz + \cdots$$

Wait a 2nd $\alpha \, dy \wedge dy \wedge dz = \beta \, dz \wedge dy \wedge dz$ because of anti-symmetric nature of ~~2-forms~~ 3-forms !

$$\Rightarrow dw = \left(\frac{\partial f_x}{\partial x} \right) dx \wedge dy \wedge dz + \left(\frac{\partial f_y}{\partial y} \right) dx \wedge dy \wedge dz +$$

$$\left(\frac{\partial f_z}{\partial z} \right) dx \wedge dy \wedge dz = \nabla \cdot \vec{f} \; dx \wedge dy \wedge dz$$

So Stokes theorem

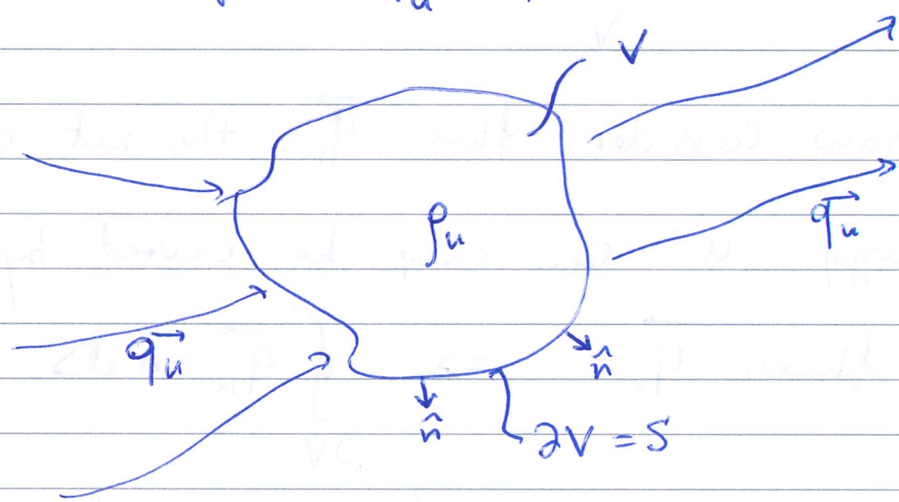
$$\int_V \nabla \cdot \vec{f}\, d^3x = \oint_S \vec{f} \cdot d\vec{S} \qquad \underline{\text{Gauss theorem}}$$

$\Rightarrow$ So the tools of vectorial calculus work with tensors, the definition of grad, div, curl is a bit different if

$$\frac{\partial g_{ij}}{\partial x^k} \neq 0 \qquad \text{though.}$$

First and eulerian approach to conservation law

Let $\rho_u(\vec{x})$ be a scalar quantity that is conserved (mass density, energy density) associated with a flux $\vec{q}_u(x)$



The total amount of "u" in volume V is at time t

$$\int_{V(t)} \rho_u(\vec{x},t)\, dV$$, at a late time $t+\Delta t$ it is

$$\int_{V(t+\Delta t)} \rho_u(\vec{x}, t+\Delta t)\, dV$$

Now, we will start with a ~~lagrangian~~ eulerian pt of view

$V(t+\Delta t) = V(t)$ fixed reference volume

The variation of mass/energy (rate of change) in V

is $\frac{d}{dt}\int_V \rho_u(\vec{x},t)\,dV$, because $V \neq V(t)$ and ref. frame is fixed

$$\frac{d}{dt}\int_V \rho_u(\vec{x},t)\,dV = \int_V \frac{\partial \rho_u}{\partial t}\,dV$$

If we now consider flux $\vec{q}_u$ the net change in mass/energy u can only be caused by unbalanced fluxes $\vec{q}_u$ $\Rightarrow$ $\oint_{\partial V} \vec{q}_u \cdot \hat{n}\,dS$

or sources/sink terms (here per unit volume)

$$\int_V \Psi\,dV$$

$$\frac{d}{dt}\int_V \rho_u(x,t)\,dV = \int_V \frac{\partial \rho_u}{\partial t}\,dV = -\oint_{\partial V} \vec{q}_u \cdot \hat{n}\,dS + \int_V \Psi\,dV$$

Gauss th:

$$\oint_{\partial V} \vec{q}_u \cdot \hat{n}\,dS = \int_V \nabla \cdot \vec{q}_u\,dV$$

$$\Rightarrow \int_V \left[\frac{\partial \rho_u}{\partial t} + \nabla \cdot \vec{q}_u - \Psi \right] dV = 0$$

true for any choice of V

$$\Rightarrow [\ldots] = 0$$

$$\Rightarrow \boxed{\frac{\partial \rho_u}{\partial t} + \nabla \cdot \vec{q}_u = \Psi}$$

Assume $\Psi = 0$ & a velocity field $\vec{V}$ carries ρ_u at flux $\vec{q}_u = \rho_u \vec{V}$

$$\Rightarrow \boxed{\frac{\partial \rho_u}{\partial t} + \nabla \cdot (\rho_u \vec{V}) = 0}$$ $\Rightarrow$ <u>continuity equation</u>