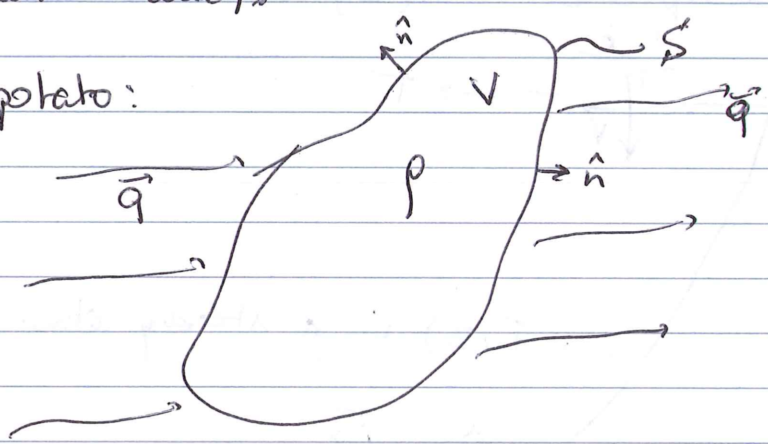
So far, we have derived a generic statement of conservation (scalar field) and retrieved the continuity equation from an Eulerian perspective (fixed volume, mass varies)

Let's approach the mass conservation from a different viewpoint, using a Lagrangian perspective (fixed mass, volume varies).

The potato:



$\vec{q}$ = mass flux

$$[\vec{q}] = \frac{kg}{m^2 s} = \rho \vec{v}$$

Mass of the "potato" at time t is

$$M(t) = \int_{V(t)} \rho(t)\, dV$$

I will define $\frac{DM}{Dt}$ as the <u>material derivative</u> of the mass with respect to time

It represents the variation of a quantity (here M) in a parcel of material and moving with it!

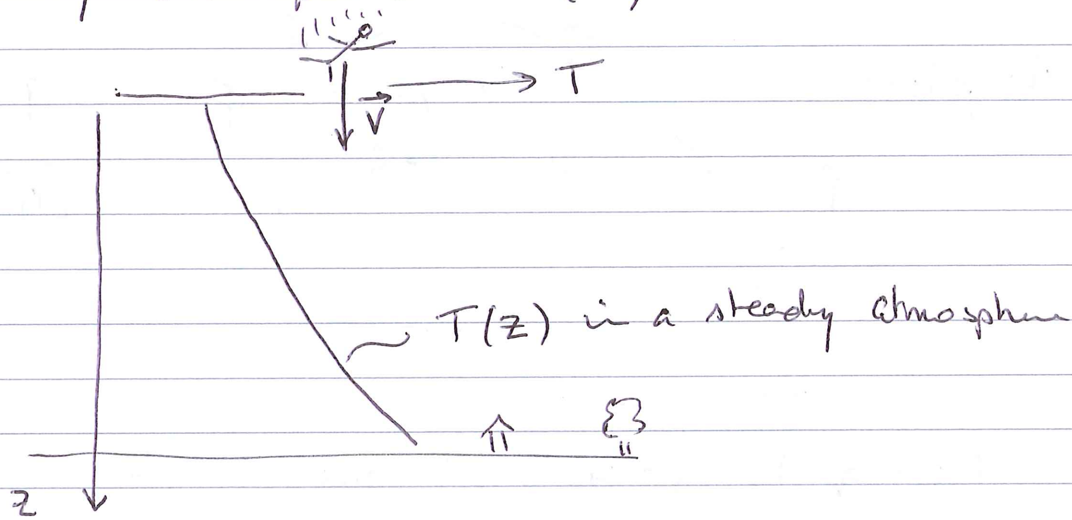
$$\frac{DM}{Dt} \neq \frac{\partial M}{\partial t}$$ in general (unless system is at rest & not deforming)

↑

non-uniform $\vec{v}(\vec{x}, t)$

In other ways, $\frac{DM}{Dt}$ tells you how much mass in the parcel of material changes as the parcel moves & deform with the "flow"/transport.

Example: The sky diver measures temperature in a steady atmosphere $T(z)$



The change in T measured by the sky diver is

$$\frac{DT}{Dt} = v\frac{\partial T}{\partial z}$$

If the T in the atmospheric is unsteady $\frac{\partial T}{\partial t} \neq 0$

then

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + v\frac{\partial T}{\partial z}$$

Lagrangian perspective

Eulerian perspective

Lagrangian mass conservation

$$\frac{DM}{Dt} = \frac{D}{Dt}\int_{V(t)} \rho(t)\, dV = 0$$

↑ why ?

In the absence of sources/sinks, there is no net flux of mass into/out the potato as it moves with $\vec{v}(\vec{x},t)$!

Now, how to deal with $\frac{D}{Dt}\int_{V(t)} \rho(t)\, dV \overset{?}{=}$

Reynolds transport theorem:

Any point $\vec{x}(t)$ within control parcel at time t can be specified by the initial position (at $t=0$) $\vec{X}_0$ & the history of the velocity field $\vec{v}(\vec{x},t)$

$$\vec{x}(t) = \vec{X}_0 + \int_0^t \vec{v}(\vec{X}_0,\tau)\, d\tau$$

then any quantity (field) $\rho(\vec{x},t)$ is actually known & only a function of time & initial position (parameter) considered $\vec{X}_0$.

$$\frac{D}{Dt}\int_{V(t)} \underbrace{\rho(\vec{x}(t),t)}_{\equiv \rho(t)}\, dV = \lim_{\delta t \to 0}\left[\frac{1}{\delta t}\left(\int_{V(t+\delta t)} \rho(t+\delta t)\, dV - \int_{V(t)} \rho(t)\, dV\right)\right]$$

Adding $\displaystyle\int_{V(t)} \rho(t+\delta t)\,dV - \int_{V(t)} \rho(t+\delta t)\,dV = 0$ we get

$$\frac{D}{D}\left[\int_{V(t)} \rho(t)\,dV\right] = \lim_{\delta t\to 0}\left\{\frac{1}{\delta t}\left[\int_{V(t+\delta t)} \rho(t+\delta t)\,dV - \int_{V(t)} \rho(t+\delta t)\,dV\right]\right.$$

$$\left. + \frac{1}{\delta t}\left[\int_{V(t)} \rho(t+\delta t)\,dV - \int_{V(t)} \rho(t)\,dV\right]\right\}$$

$$= \int_{V(t)} \frac{\partial \rho}{\partial t}\,dV$$

$$\lim_{\delta t\to 0}\left\{\frac{1}{\delta t}\left[\int_{V(t+\delta t)} \rho(t+\delta t)\,dV - \int_{V(t)} \rho(t+\delta t)\,dV\right]\right\} = \lim_{\delta t\to 0} \frac{1}{\delta t}\int_{V(t+\delta t)-V(t)} \rho(t+\delta t)\,dV$$

What is $V(t+\delta t) - V(t)$? if $S(t)$ is bounding surface around $V(t)$, elements of surface travels a distance

$\vec{v}\cdot\hat{n}\,\delta t$ from $t \to t+\delta t$

$$dV = d(V(t+\delta t) - V(t)) = \vec{v}\cdot\hat{n}\,\delta t\,dS$$

$$\int_{V(t+\delta t)-V(t)} \rho(t+\delta t)\, dV = \oint_{S} \rho(t+\delta t)\, \vec{v}\cdot\hat{n}\, dS\, \delta t$$

So
$$\lim_{\delta t\to 0} \frac{1}{\delta t}\int_{V(t+\delta t)-V(t)} \rho(t+\delta t)\, dV = \lim_{\delta t\to 0} \oint_{S(t)} \rho(t+\delta t)\,\vec{v}\cdot\hat{n}\, dS$$

$$= \oint_{S(t)} \rho(t)\,\vec{v}\cdot\hat{n}\, dS$$

$$\Rightarrow \boxed{\frac{D}{Dt}\int_{V(t)} \rho(t)\, dV = \int_{V(t)} \frac{\partial \rho}{\partial t}\, dV + \underbrace{\oint_{S} \rho\,\vec{v}\cdot\hat{n}\, dS}_{=\int_V \nabla\cdot(\rho\vec{v})\, dV}}$$

$$\Rightarrow \frac{DM}{Dt} = \frac{D}{Dt}\int_{V(t)} \rho\, dV = \int_{V(t)} \left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v})\right] dV = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0 \qquad \text{(continuity eq)}$$

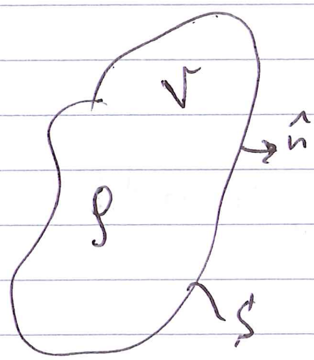
Now, by definition of material derivative

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \vec{v}\cdot\nabla\rho = \underbrace{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v})}_{=0} + \vec{v}\cdot\nabla\rho - \nabla\cdot(\rho\vec{v})$$

$$\frac{D\rho}{Dt} = \vec{V}\cdot\nabla\rho - \nabla(\rho\vec{V}) = \cancel{\vec{V}\cdot\nabla\rho} - (\cancel{\vec{V}\cdot\nabla\rho} + \rho\nabla\cdot\vec{V})$$
$$= \rho\nabla\cdot\vec{V}$$

If a fluid is incompressible $\frac{D\rho}{Dt} = 0 \Rightarrow \boxed{\nabla\cdot\vec{V} = 0}$

Conservation of linear momentum



Newton's law (2^{nd})

Change in momentum = $\sum$ forces exerted on it (body)

Momentum at time t

$\int_{V(t)} \rho\vec{V}\,dV$ → change in momentum acting on a parcel of material

$$\frac{D}{Dt}\int_{V(t)} \rho\vec{V}\,dV = \sum \overrightarrow{\text{forces}}$$

2 types of forces → body forces (act on whole body)

→ stress acting on surface

Let's define the stress tensor $\underline{\underline{T}}$ (2^{nd} rank)

and the stress vector acting on an oriented surface

$$\vec{t} = \underline{\underline{T}} \cdot \hat{n}$$

$$\frac{D}{Dt}\int_{V(t)} \rho \vec{v}\, dV = \int_{V(t)} \underbrace{\rho \vec{g}}_{\text{e.g. body force}}\, dV + \oint_{S} \vec{t}\, dS$$

$$\frac{D}{Dt}\int_{V(t)} \rho \vec{v}\, dV = \int_{V(t)} \rho \vec{g}\, dV + \oint_{S} \underline{\underline{T}} \cdot \hat{n}\, dS \overset{\text{divergence th.}}{=} \int_{V(t)} \left[\rho \vec{g} + \nabla \cdot \underline{\underline{T}}\right] dV$$

Using Reynolds transport theorem (works with vectors)

$$\frac{D}{Dt}\int_{V(t)} \rho \vec{v}\, dV = \int_{V(t)} \left[\frac{\partial \rho \vec{v}}{\partial t} + \nabla \cdot (\rho \vec{v}\vec{v})\right] dV$$

$$\Rightarrow \int_{V(t)} \left[\frac{\partial \rho \vec{v}}{\partial t} + \nabla \cdot (\rho \vec{v}\vec{v}) - \rho \vec{g} - \nabla \cdot \underline{\underline{T}}\right] dV = 0$$

True for any $V(t)$ so

$$\frac{\partial \rho \vec{v}}{\partial t} + \nabla \cdot (\rho \vec{v}\vec{v}) = \rho \vec{g} + \nabla \cdot \underline{\underline{T}}$$

In linear elasticity $\rho \vec{v}\vec{v}$ is very small & neglected

$$\Rightarrow \frac{\partial \rho \vec{v}}{\partial t} = \rho \vec{g} + \nabla \cdot \underline{\underline{T}}$$

$\rho\vec{v}\vec{v}$ is sometimes referred to as Reynolds stress.

$$\nabla\cdot(\rho\vec{v}\vec{v}) = \rho\vec{v}\cdot\nabla\vec{v} + \vec{v}\,\nabla\cdot(\rho\vec{v})$$

So LHS

$$\frac{\partial \rho\vec{v}}{\partial t} + \nabla\cdot(\rho\vec{v}\vec{v}) = \rho\frac{\partial\vec{v}}{\partial t} + \vec{v}\underbrace{\left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v})\right]}_{=0} + \rho\vec{v}\cdot\nabla\vec{v}$$

$$= \rho\frac{\partial\vec{v}}{\partial t} + \rho\vec{v}\cdot\nabla\vec{v}$$

$\Rightarrow$ Cauchy's equation

$$\boxed{\rho\left[\frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}\right] = \nabla\cdot\underline{\underline{T}} + \rho\vec{g}}$$

Again for elastic (linear) solid

$$\rho\frac{\partial\vec{v}}{\partial t} \approx \nabla\cdot\underline{\underline{T}} + \rho\vec{g}$$

Constitutive equations:

What is $\underline{\underline{T}}$? , logically it should depend on deformation, pressure ...

Let's start with an isotropic solid (elastic):

$$\underline{\underline{T}} = \underbrace{a_0 \, \mathbb{1}}_{\substack{\text{scalar} \\ \text{contribution} \\ \text{is isotropic}}} + a_1 \underline{\underline{B_1}} + a_2 \underline{\underline{B_1}}^2$$

$\underline{\underline{B_1}}$: 2nd rank tensor associated with deformation

Material objectivity = we want the constitutive laws to obey coordinate invariance (tensorial law)

Material symmetry = Constitutive law should portray the symmetries of the material (isotropic solid → invariance with orthogonal transformations (rotations))

⇒ tensor $\underline{\underline{B_1}}$ has to be symmetric to satisfy these 2 conditions. If displacement vector is $\vec{u}$ (velocity $= \frac{\partial \vec{u}}{\partial t}$), then $\nabla \vec{u}$ is a 2nd rank object (deformation gradient) but it is not transforming in an invariant way

but $\underline{\underline{\varepsilon}} = \frac{1}{2}\left[\nabla\vec{u} + \nabla\vec{u}^T\right]$ does

strain - tensor

$\underline{\underline{B}}_1 = f(\underline{\underline{\varepsilon}})$, then from isotropy / symmetries

=>

$$a_1 \underline{\underline{B}}_1 = \underset{\text{elastic moduli}}{\underline{\underline{\underline{\underline{C}}}}} : \underline{\underline{\varepsilon}}$$

or index-wise $a_1 B_{ij} = C_{ijkl}\,\varepsilon_{kl}$

& only 2 independent material properties

λ, μ (Lamé coefficients) arise from $\underline{\underline{\underline{\underline{C}}}}$

Linear elasticity (limits of small deformation)

$a_2 \underline{\underline{B}}_1^2 \longrightarrow$ small & neglected.

What about a Newtonian fluid? $\underline{\underline{T}} = ?$

$$\underline{\underline{T}} = a_0 \mathbb{1} + a_1 \underline{\underline{D}}$$

stop at linear term in $\underline{\underline{D}}$ (related to deformation rate)

$\underline{\underline{D}} \neq \underline{\underline{B}}$ for elastic because:

(i) A fluid deforms diffently from a "rigid" solid body $\Rightarrow$ different material symmetry.

Example: take a bucket with water at rest (initial state), stir it and let it go back to rest ... the configuration at the end is indistinguishable from the initial one, however, very likely most molecules are distributed very differently. Not true for rigid body in the limit of small deformations.

$\Rightarrow$ For a fluid it is not the diffence between the initial & final configuration that matters but the rate of change (time matters)

So $\underline{\underline{D}}$ needs to reflect rate of deformation

$\nabla \vec{u} \longrightarrow \nabla \vec{v}$ would be more appropriate

$$\left(\vec{v} = \frac{\partial \vec{u}}{\partial t} \right)$$

However $\nabla \vec{v}$ doesn't transform properly (tensor)

but its symmetric part does $\underline{\underline{\dot{\varepsilon}}} = \frac{1}{2} \left(\nabla \vec{v} + \nabla \vec{v}^{T} \right)$

strain-rate

What is a_0?

Fluid at rest, Cauchy's equation reduces to

$$0 = \nabla \cdot \underline{\underline{T}} + \rho \vec{g} \qquad \text{(linear mechanical eq.)}$$

At rest, the only stress acting is pressure, which acts normally to a surface

$$\underline{t} = \hat{n} \cdot \underline{\underline{T}} = -\hat{n}\, p$$

$$\Rightarrow \underline{\underline{T}} = -p\,\underline{\mathbb{1}} \qquad \Rightarrow a_0 = -p$$

$$\underline{\underline{T}} = -p\,\underline{\mathbb{1}} + \underline{\underline{\tau}}$$, $\underline{\underline{\tau}}$ is deviatoric stress ($\sim \underline{\underline{\dot{\varepsilon}}}$)

Isotropic fluids & symmetries $\rightarrow \lambda, \mu$

(equivalent to Lamé's parameters)

$$\underline{\underline{T}} = -p\,\underline{\mathbb{1}} + \lambda \operatorname{tr} \underline{\underline{\dot{\varepsilon}}}\, \underline{\mathbb{1}} + 2\mu\, \underline{\underline{\dot{\varepsilon}}}$$

λ = bulk viscosity

μ = dynamic viscosity

& $\operatorname{tr} \underline{\underline{\dot{\varepsilon}}} = \nabla \cdot \vec{V}$ (= 0 if incompressible)